

TWO-SIDED BGG RESOLUTIONS OF ADMISSIBLE REPRESENTATIONS

TOMOYUKI ARAKAWA

ABSTRACT. We prove the conjecture of Frenkel, Kac and Wakimoto [FKW] on the existence of two-sided BGG resolutions of G -integrable admissible representations of affine Kac-Moody algebras at fractional levels.

1. INTRODUCTION

Wakimoto modules are representations of non-twisted affine Kac-Moody algebras introduced by Wakimoto [Wak] in the case of $\widehat{\mathfrak{sl}}_2$ and by Feigin and Frenkel [FF1] in the general case. Wakimoto modules have useful applications in representation theory and conformal field theory. In these applications it is important to have a resolution of an irreducible highest weight representation $L(\lambda)$ of an affine Kac-Moody algebra \mathfrak{g} in terms of Wakimoto modules, that is, a complex

$$C^\bullet(\lambda) : \rightarrow C^{i-1}(\lambda) \xrightarrow{d_{i-1}} C^i(\lambda) \xrightarrow{d_i} C^{i+1}(\lambda) \rightarrow \dots$$

with a differential d_i which is a \mathfrak{g} -module homomorphism such that $C^i(\lambda)$ is a direct sum of Wakimoto modules and

$$H^i(C^\bullet(\lambda)) = \begin{cases} L(\lambda) & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The existence of such a resolution has been proved by Feigin and Frenkel [FF2] for any integrable representations over arbitrary \mathfrak{g} and by Bernard and Felder [BF] and Feigin and Frenkel [FF2] for any admissible representation [KW2] over $\widehat{\mathfrak{sl}}_2$. In their study of W -algebras Frenkel, Kac and Wakimoto [FKW, Conjecture 3.5.1] conjectured the existence of such a resolution for any principle admissible representations over arbitrary \mathfrak{g} . In this paper we prove the existence of a two-sided resolution of in terms of Wakimoto modules for any $\mathring{\mathfrak{g}}$ -integrable admissible representations over arbitrary \mathfrak{g} (Theorem 6.10), where $\mathring{\mathfrak{g}}$ is the classical part of \mathfrak{g} . For a general principal admissible representation of \mathfrak{g} we obtain the two-sided resolution in terms of twisted Wakimoto modules (Theorem 6.14).

Let us sketch the proof of our result briefly. By Fiebig's equivalence [Fie] the block of the category \mathcal{O} of \mathfrak{g} containing an admissible representation $L(\lambda)$ is equivalent to the block containing an integrable representation¹. Therefore an admissible representation admit a usual BGG type resolution in terms of Verma modules by the result of [GL, RCW]. Hence the idea of Arkhipov [Ark1] is applicable in our

This work is partially supported by the JSPS Grant-in-Aid for Scientific Research (B) No. 20340007 and JSPS Grant-in-Aid for Challenging Exploratory Research No. 23654006.

¹In the case $L(\lambda)$ is a non-principal G -integrable admissible representation this is a block of another Kac-Moody algebra.

situation: One can obtain a twisted BGG resolution of $L(\lambda)$ in terms of twisted Verma modules by applying the twisting functor T_w [Ark1] to the BGG resolution of $L(\lambda)$ as we have the “Borel-Weil-Bott” vanishing property [AS]

$$\mathcal{L}_i T_w L(\lambda) \cong \begin{cases} L(\lambda) & \text{if } i = \ell_\lambda(w), \\ 0 & \text{otherwise} \end{cases}$$

for $w \in \mathcal{W}(\lambda)$, where $\mathcal{W}(\lambda)$ is the integral Weyl group of λ and $\ell_\lambda : \mathcal{W}(\lambda) \rightarrow \mathbb{Z}_{\geq 0}$ is the length function, see Theorem 5.19. It remains to show that one can construct an inductive system of twisted BGG resolutions $\{B_w^\bullet(\lambda)\}$ of $L(\lambda)$ such that the complex $\varinjlim_w B_w^\bullet(\lambda)$ gives the required two-sided resolution of $L(\lambda)$, see §6 for the details.

We note that by applying the (generalized) quantum Drinfeld-Sokolov reduction functor [FKW, KRW] to the (duals of the) two-sided BGG resolutions of admissible representations we obtain resolutions of some of simple modules over W -algebras in terms of free field realizations due to the vanishing of cohomology [A1, A2, A3, A4, A5]. In particular we obtain two-sided resolutions of all the minimal series representations [FKW] of the W -algebras associated with principal nilpotent elements in terms of free bosonic realizations.

This paper is organized as follows. In §2 we collect and prove some basic results about semi-infinite cohomology [Fei] and semi-regular bimodules [Vor1] which are needed for later use. In §2 we collect basic results on the semi-infinite Bruhat ordering of an affine Weyl group defined by Lusztig [Lus] and study the semi-infinite analogue of parabolic subgroups. Semi-infinite Bruhat ordering is important for us since it describes the space of homomorphisms between Wakimoto modules. The semi-infinite analogue of parabolic subgroups is important for the semi-infinite restriction functors studied in §7. In §4 we define Wakimoto modules and twisted Verma modules following [Vor2] and study some of their basic properties. In particular we prove the uniqueness of Wakimoto modules which was stated in [FF2] without a proof (Theorem 4.6). In §5 we study the twisting functor for Kac-Moody algebras. In particular we generalize the Borel-Weil-Bott vanishing property proved in [AS]. In §6 we prove the main results of this paper. In §7 we study the semi-infinite restriction functor and establish the semi-infinite analogue of the generalized Borel-Weil theorem [Kos] for minimal parabolic subalgebras (Theorem 7.8). This is a non-trivial fact since admissible representations are not unitarizable unless they are integrable. Based on the results obtained in this section in a subsequent paper [A6] we prove the conjecture of Admović and Milas [AM] on the rationality of admissible affine vertex algebras in the category \mathcal{O} in full generality.

Acknowledgments. Some part of this work was done while the author was visiting Weizmann Institute, Israel, in May 2011, Emmy Noether Center in Erlangen, Germany in June 2011, Isaac Newton Institute for Mathematical Sciences, UK, in 2011, The University of Manchester, University of Birmingham, The University of Edinburgh, Lancaster University, York University, UK, in November 2011, Academia Sinica, Taiwan, in December 2011. He is grateful to those institutes for their hospitality.

2. SEMI-REGULAR BIMODULES AND SEMI-INFINITE COHOMOLOGY

2.1. Some notation. For \mathbb{Z} -graded vector spaces $M = \bigoplus_{n \in \mathbb{Z}} M_n$, $N = \bigoplus_{n \in \mathbb{Z}} N_n$ let

$$\begin{aligned} \mathcal{H}om_{\mathbb{C}}(M, N) &= \bigoplus_{n \in \mathbb{Z}} \mathcal{H}om_{\mathbb{C}}(M, N)_n, \\ \mathcal{H}om_{\mathbb{C}}(M, N)_n &= \{f \in \text{Hom}_{\mathbb{C}}(M, N); f(M_i) \subset N_{i+n}\}, \\ \mathcal{E}nd_{\mathbb{C}}(M) &= \mathcal{H}om_{\mathbb{C}}(M, M). \end{aligned}$$

We denote by $M^* = \bigoplus_{n \in \mathbb{Z}} (M^*)_n$ the space $\mathcal{H}om_{\mathbb{C}}(M, \mathbb{C})$, where \mathbb{C} is considered as a graded vector space concentrated in the degree 0 component. If M, N are module over an algebra A we denote by $\mathcal{H}om_A(M, N)$ the space of all A -homomorphisms in $\mathcal{H}om_{\mathbb{C}}(M, N)$.

2.2. Semi-infinite structure. Let \mathfrak{g} be a complex Lie algebra. A *semi-infinite structure* [Vor1] of \mathfrak{g} is the following data:

- (i) a \mathbb{Z} -grading $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$ of \mathfrak{g} with finite-dimensional homogeneous components, $\dim_{\mathbb{C}} \mathfrak{g}_n < \infty$ for all n ,
- (ii) a *semi-infinite 1-cochain* $\gamma : \mathfrak{g} \rightarrow \mathbb{C}$.

Here by a semi-infinite 1-cochain we mean the following: Decompose \mathfrak{g} into the direct sum of two subalgebras

$$\begin{aligned} \mathfrak{g} &= \mathfrak{g}_+ \oplus \mathfrak{g}_-, \\ \mathfrak{g}_+ &= \bigoplus_{i \geq 0} \mathfrak{g}_i, \quad \mathfrak{g}_- = \bigoplus_{i < 0} \mathfrak{g}_i. \end{aligned}$$

A linear map $\gamma : \mathfrak{g} \rightarrow \mathbb{C}$ is called a semi-infinite 1-cochain if γ satisfies

$$\gamma([x, y]) = \text{tr}((\text{ad } x)_{+-}(\text{ad } y)_{-+} - (\text{ad } y)_{+-}(\text{ad } x)_{-+}) \quad \text{for } x, y \in \mathfrak{g},$$

where $(\text{ad } x)_{\pm\mp}$ denotes the composition $\mathfrak{g}_{\mp} \xrightarrow{\text{ad } x} \mathfrak{g} \xrightarrow{\text{projection}} \mathfrak{g}_{\pm}$.

In the rest of this section we assume that \mathfrak{g} is equipped with a semi-infinite structure such that $\gamma(\sum_{i \neq 0} \mathfrak{g}_i) = 0$.

We denote by U, U_-, U_+ , the enveloping algebras of $\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-$ by respectively. These algebras inherit a \mathbb{Z} -grading from the corresponding Lie algebras.

Let $\hat{\mathcal{O}}^{\mathfrak{g}}$ be the category of \mathbb{Z} -graded \mathfrak{g} -modules $M = \bigoplus_{n \in \mathbb{Z}} M_n$ with $\dim M_n < \infty$ for all m which are locally finite for \mathfrak{g}_+

2.3. Semi-infinite cohomology. Choose a basis $\{x_i; i \in \mathbb{Z}\}$ of \mathfrak{g} such that $\{x_i; i \geq 0\}$ and $\{x_i; i < 0\}$ are a basis of \mathfrak{g}_+ and \mathfrak{g}_- , respectively, and let $\{c_{ij}^k\}$ be the structure constant: $[x_i, x_j] = \sum_k c_{ij}^k x_k$.

Denote by $\mathcal{Cl}(\mathfrak{g})$ the Clifford algebra associated with $\mathfrak{g} \oplus \mathfrak{g}^*$, which has the following generators and relations:

$$\begin{aligned} \text{generators: } & \psi_i, \psi_i^* \quad (i \in \mathbb{Z}), \\ \text{relations: } & \{\psi_i, \psi_j^*\} = \delta_{i,j}, \quad \{\psi_i, \psi_j\} = \{\psi_i^*, \psi_j^*\} = 0. \end{aligned}$$

The space of the semi-infinite forms $\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g})$ of \mathfrak{g} is by definition the irreducible representation of $\mathcal{Cl}(\mathfrak{g})$ generated by the vector $\mathbf{1}$ satisfying

$$\psi_i \mathbf{1} = 0 \quad (i \geq 0), \quad \psi_i^* \mathbf{1} = 0 \quad (i > 0).$$

It is graded by $\deg \mathbf{1} = 0$, $\deg \psi_i^* = 1$ and $\deg \psi_i = -1$: $\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g}) = \bigoplus_{p \in \mathbb{Z}} \bigwedge^{\frac{\infty}{2}+p}(\mathfrak{g})$.

For $A \in \text{End}_{\mathbb{C}}(\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g}))$ of degree n set

$$: \psi_k A := \begin{cases} : \psi_k A & \text{if } k < 0, \\ (-1)^n : A \psi_k : & \text{if } k \geq 0. \end{cases}$$

Then the following defines a \mathfrak{g} -module structure on $\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g})$:

$$(1) \quad x_i \mapsto : \text{ad}(x_i) : + \gamma(x_i),$$

where

$$: \text{ad } x_i := \sum_{j,k} c_{ij}^k : \psi_k \psi_j^* :.$$

For $M \in \tilde{\mathcal{O}}^{\mathfrak{g}}$, define $d \in \text{End}(M \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g}))$ by

$$d = \sum_i x_i \otimes \psi_i^* - 1 \otimes \frac{1}{2} \sum_{i,j,k} c_{ij}^k : \psi_i (: \psi_j \psi_k^* :) : + 1 \otimes \sum_i \gamma(x_i) \psi_i^*$$

Then

$$d^2 = 0, \quad d(M \otimes \bigwedge^{\frac{\infty}{2}+p}(\mathfrak{g})) \subset M \otimes \bigwedge^{\frac{\infty}{2}+p+1}(\mathfrak{g}).$$

The cohomology of the complex $(M \otimes \bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g}), d)$ is called the *semi-infinite \mathfrak{g} -cohomology* with coefficients in M and denoted by $H^{\frac{\infty}{2}+\bullet}(\mathfrak{g}, M)$ ([Feř, Vor1]).

2.4. Semi-regular bimodules. We consider the graded dual U_+^* as an \mathfrak{g}_+ -bimodule by $(Xf)(n) = f(nX)$, $(fX)(n) = f(Xn)$ for $f \in U_+^*$, $X \in \mathfrak{g}_+$, $n \in U_+$.

Define

$$(2) \quad SS\mathfrak{g} := U_+^* \otimes_{U_+} U.$$

Then $SS\mathfrak{g}$ is naturally a left \mathfrak{g}_+ -module and a right \mathfrak{g} -module. It is known [Vor1, Soe2, Vor2] that it admits a \mathfrak{g} -bimodule structure : Consider the linear space isomorphism

$$SS\mathfrak{g} \cong U_+^* \otimes_{\mathbb{C}} U_- \cong \text{Hom}_{\mathbb{C}}(U_+, U_-) \cong \text{Hom}_{U_-}(U, U_- \otimes_{\mathbb{C}} \mathbb{C}_{-\gamma}).$$

The left \mathfrak{g} -module structure of $SS\mathfrak{g}$ is defined though the isomorphism $SS\mathfrak{g} \cong \text{Hom}_{U_-}(U, U_- \otimes_{\mathbb{C}} \mathbb{C}_{-\gamma})$. The bimodule $SS\mathfrak{g}$ is called the *semi-regular bimodule* of \mathfrak{g} . We have

$$SS\mathfrak{g} \cong U \otimes_{U_+} U_+^*$$

as left U -modules and right U_+ -modules.

Let M be a \mathfrak{g} -module and consider the following four left \mathfrak{g} -module structures on $SS\mathfrak{g} \otimes_{\mathbb{C}} M$:

$$(3) \quad \pi_1(X)(s \otimes m) = -(sX) \otimes m + s \otimes Xm, \quad \pi_2(X)(s \otimes m) = (Xs) \otimes m,$$

$$(4) \quad \pi'_1(X)(s \otimes m) = -(sX) \otimes m, \quad \pi'_2(X)(s \otimes m) = (Xs) \otimes m + s \otimes (Xm),$$

for $X \in \mathfrak{g}$, $s \in SS\mathfrak{g}$, $m \in M$. Clearly, the two actions π_1 and π_2 (resp. π'_1 and π'_2) commute.

Proposition 2.1 (cf. [AG, 6.4]). *For $M \in \tilde{\mathcal{O}}^{\mathfrak{g}}$ the two U -bimodule structures (π_1, π_2) and (π'_1, π'_2) on $SS\mathfrak{g} \otimes_{\mathbb{C}} M$ are equivalent. Namely there exists a linear isomorphism $\Phi : SS\mathfrak{g} \otimes_{\mathbb{C}} M \xrightarrow{\sim} SS\mathfrak{g} \otimes_{\mathbb{C}} M$ such that $\Phi \circ \pi'_i(X) = \pi_i(X) \circ \Phi$ for $i = 1, 2$, $X \in \mathfrak{g}$.*

Proof. Consider the graded dual $M^* = \bigoplus_n (M^*)_n$ as a graded right \mathfrak{g} -module by the action $(Xf)(m) = f(Xm)$. We have

$$SS\mathfrak{g} \otimes_{\mathbb{C}} M \cong \text{Hom}_{U_-}(U, U_- \otimes_{\mathbb{C}} \mathbb{C}_{-\gamma}) \otimes_{\mathbb{C}} M \cong \text{Hom}_{U_-}(U \otimes_{\mathbb{C}} M^*, U_- \otimes_{\mathbb{C}} \mathbb{C}_{-\gamma}).$$

Let $\Psi_1 : SS\mathfrak{g} \otimes M \xrightarrow{\sim} SS\mathfrak{g} \otimes M$ be the linear map induced from the linear isomorphism

$$\begin{array}{ccc} \text{Hom}_{U_-}(U \otimes_{\mathbb{C}} M^*, U_- \otimes_{\mathbb{C}} \mathbb{C}_{-\gamma}) & \xrightarrow{\sim} & \text{Hom}_{U_-}(U \otimes_{\mathbb{C}} M^*, U_- \otimes_{\mathbb{C}} \mathbb{C}_{-\gamma}), \\ f & \mapsto & (u \otimes g \mapsto f((1 \otimes g)\Delta(u))), \end{array}$$

and the above identification, where $\Delta : U \rightarrow U \otimes_{\mathbb{C}} U$ is the coproduct. Then we have

$$\begin{aligned} \Psi_1 \circ \pi'_2(X) &= \pi_2(X) \circ \Psi_1 \quad \text{for } X \in \mathfrak{g}, \\ \Psi_1 \circ \pi'_1(X) &= \pi'_1(X) \circ \Psi_1 \quad \text{for } X \in \mathfrak{g}_-. \end{aligned}$$

Next let $\Psi_2 : SS\mathfrak{g} \otimes_{\mathbb{C}} M \xrightarrow{\sim} SS\mathfrak{g} \otimes_{\mathbb{C}} M$ be the linear map induced from the linear isomorphism

$$\begin{array}{ccc} U_+^* \otimes_{\mathbb{C}} U_- \otimes_{\mathbb{C}} M & \xrightarrow{\sim} & U_+^* \otimes_{\mathbb{C}} U_- \otimes_{\mathbb{C}} M, \\ f \otimes u \otimes m & \mapsto & f \otimes ((1 \otimes m)\Delta(u)), \end{array}$$

where M is considered as a right \mathfrak{g} -module by the action $mX = -Xm$. Then we have

$$\begin{aligned} \Psi_2 \circ \pi'_1(X) &= \pi_1(X) \circ \Psi_2 \quad \text{for } X \in \mathfrak{g}_-, \\ \Psi_2 \circ \pi_2(X) &= \pi_2(X) \circ \Psi_2 \quad \text{for } X \in \mathfrak{g}. \end{aligned}$$

Let $\Phi = \Psi_2 \circ \Psi_1$. It remains to show that $\Phi \circ \pi'_1(X) = \pi_1(X) \circ \Phi$ for $X \in \mathfrak{g}_+$. But from the definition it follows that

$$\Phi|_{U_+^* \otimes M} \circ \pi'_1(X) = \pi_1(X) \circ \Phi|_{U_+^* \otimes M},$$

where $\Phi|_{U_+^* \otimes M}$ is the restriction of Φ to the subspace $U_+^* \otimes_{\mathbb{C}} \mathbb{C} \otimes_{\mathbb{C}} M \subset U_+^* \otimes_{\mathbb{C}} U_- \otimes_{\mathbb{C}} M = SS\mathfrak{g} \otimes_{\mathbb{C}} M$. Since $SS\mathfrak{g} \otimes M$ is generated by $U_+^* \otimes_{\mathbb{C}} M$ as over U_- by the action π_2 or π'_2 , the commutativity of π_1 with π_2 and π'_1 with π'_2 implies the required property. \square

2.5. Semi-infinite induction. Let $\mathfrak{h} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{h}_n$ be a graded Lie subalgebra of \mathfrak{g} such that $\gamma|_{\mathfrak{h}}$ is a semi-infinite 1-cochain of \mathfrak{h} . Following [Vor2] we define the *semi-induced \mathfrak{g} -module* $\text{S-ind}_{\mathfrak{h}}^{\mathfrak{g}} M$ as

$$\text{S-ind}_{\mathfrak{h}}^{\mathfrak{g}} M := H^{\frac{\infty}{2}+0}(\mathfrak{h}, SS\mathfrak{g} \otimes_{\mathbb{C}} M),$$

where $SS\mathfrak{g} \otimes_{\mathbb{C}} M$ is considered as an \mathfrak{h} -module by the action π_1 defined in (3). The space $\text{S-ind}_{\mathfrak{h}}^{\mathfrak{g}} M$ inherits the structure of a \mathfrak{g} -module from the action π_2 in (3).

Lemma 2.2. *The assignment $\text{S-ind}_{\mathfrak{h}}^{\mathfrak{g}} : M \mapsto \text{S-ind}_{\mathfrak{h}}^{\mathfrak{g}} M$ defines an exact functor from $\tilde{\mathcal{O}}^{\mathfrak{h}}$ to $\tilde{\mathcal{O}}^{\mathfrak{g}}$.*

Proof. It is obvious that $\text{S-ind } M \in \tilde{\mathcal{O}}^{\mathfrak{g}}$ since $SS\mathfrak{g} \otimes_{\mathbb{C}} M \in \mathcal{O}^{\mathfrak{g}}$. By Proposition 2.1 we may replace the actions of π_1 and π_2 on $SS\mathfrak{g} \otimes_{\mathbb{C}} M$ with π'_1 and π'_2 , respectively. Hence

$$(5) \quad H^{\frac{\infty}{2}+\bullet}(\mathfrak{h}, SS\mathfrak{g} \otimes_{\mathbb{C}} M) \cong H^{\frac{\infty}{2}+\bullet}(\mathfrak{h}, SS\mathfrak{g}) \otimes_{\mathbb{C}} M.$$

Since $SS\mathfrak{g}$ is in particular free over \mathfrak{h}_- and cofree over \mathfrak{h}_+ , we have $H^{\frac{\infty}{2}+i}(\mathfrak{h}, SS\mathfrak{g}) = 0$ for $i \neq 0$ by [Vor1, Theorem 2.1]. This proves that the functor $\text{S-ind}_{\mathfrak{h}}^{\mathfrak{g}}$ is exact. \square

In the case that $\mathfrak{h} = \mathfrak{g}$ and $\gamma_0 = \gamma$, we have the following assertion.

Proposition 2.3 ([Vor2, (1.9)]). *The functor $\mathrm{S}\text{-ind}_{\mathfrak{g}}^{\mathfrak{g}} : \tilde{\mathcal{O}}^{\mathfrak{g}} \rightarrow \tilde{\mathcal{O}}^{\mathfrak{g}}$ is isomorphic to the identity functor.*

Proof. As $H^{\frac{\infty}{2}+0}(\mathfrak{g}, SS\mathfrak{g})$ is isomorphic to the trivial representation \mathbb{C} of \mathfrak{g} ([Vor1, Theorem 2.1]) (5) gives that $\mathrm{S}\text{-ind}_{\mathfrak{g}}^{\mathfrak{g}} M \cong M$ as required. \square

2.6. Suppose that \mathfrak{g} admits a decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \bar{\mathfrak{a}}$$

with graded subalgebras \mathfrak{a} and $\bar{\mathfrak{a}}$ such that $\gamma|_{\mathfrak{a}}$ and $\gamma|_{\bar{\mathfrak{a}}}$ are semi-infinite 1-cochains of \mathfrak{a} and $\bar{\mathfrak{a}}$, respectively.

Lemma 2.4. *$SS\mathfrak{g} \cong SS\mathfrak{a} \otimes_{\mathbb{C}} SS\bar{\mathfrak{a}}$ as left \mathfrak{a} -modules and right $\bar{\mathfrak{a}}$ -modules.*

Proof. First, the multiplication map

$$\varphi : U(\mathfrak{a}_+) \otimes_{\mathbb{C}} U_+^* \otimes_{\mathbb{C}} U(\bar{\mathfrak{a}}_+) \rightarrow SS\mathfrak{g}$$

is surjective since the image is a $U(\mathfrak{g})$ -bi-submodule of $SS\mathfrak{g}$. Hence it is an isomorphism by the quality of the graded dimensions.

Next, since $U_+ \cong U(\bar{\mathfrak{a}}_+) \otimes_{\mathbb{C}} U(\mathfrak{a}_+)$ as left $\bar{\mathfrak{a}}_+$ -modules and right \mathfrak{a}_+ -modules, $U_+^* \cong U(\mathfrak{a}_+)^* \otimes_{\mathbb{C}} U(\bar{\mathfrak{a}}_+)^*$ as left \mathfrak{a}_+ -modules and right $\bar{\mathfrak{a}}_+$ -modules. The left $U(\mathfrak{a})$ -submodule generated by $U(\mathfrak{a}_+)^* \otimes_{\mathbb{C}} \mathbb{C} \subset U_+^* \subset SS\mathfrak{g}$ is isomorphic to $SS\mathfrak{a}$, and the right $U(\bar{\mathfrak{a}})$ -submodule $\mathbb{C} \otimes U(\bar{\mathfrak{a}}_+)^* \subset SS\mathfrak{g}$ is isomorphic to $SS\bar{\mathfrak{a}}$. It follows that

$$\begin{aligned} SS\mathfrak{a} \otimes_{\mathbb{C}} SS\bar{\mathfrak{a}} &\cong (U(\mathfrak{a}) \otimes_{U(\mathfrak{a})_+} U(\mathfrak{a}_+)^*) \otimes_{\mathbb{C}} (U(\bar{\mathfrak{a}}_+)^*_{U(\mathfrak{a}_+)} U(\mathfrak{a})) \\ &\xrightarrow{\sim} U(\mathfrak{a}) \otimes_{U(\mathfrak{a})_+} U_+^* \otimes_{U(\bar{\mathfrak{a}}_+)} U(\bar{\mathfrak{a}}_+) \xrightarrow{\varphi} SS\mathfrak{g}. \end{aligned}$$

\square

Lemma 2.5. *For $M \in \tilde{\mathcal{O}}^{\bar{\mathfrak{a}}}$, $\mathrm{S}\text{-ind}_{\bar{\mathfrak{a}}}^{\bar{\mathfrak{a}}} M \cong SS\mathfrak{a} \otimes_{\mathbb{C}} M$ as \mathfrak{a} -modules, where \mathfrak{a} acts only on the first factor of $SS\mathfrak{a} \otimes_{\mathbb{C}} M$.*

Proof. We have $H^{\frac{\infty}{2}+i}(\bar{\mathfrak{a}}, SS\mathfrak{g} \otimes_{\mathbb{C}} M) \cong SS\mathfrak{a} \otimes H^{\frac{\infty}{2}+i}(\bar{\mathfrak{a}}, SS\bar{\mathfrak{a}} \otimes M) \cong SS\mathfrak{a} \otimes M$ by Lemmas 2.3 and 2.4. \square

3. SEMI-INFINITE BRUHAT ORDERING

3.1. Affine Kac-Moody algebras and affine Weyl groups. We first fix some notation which are used for the rest of the paper.

Let $\mathring{\mathfrak{g}}$ be a complex simple Lie algebra, and fix a Cartan subalgebra $\mathring{\mathfrak{h}}$ of $\mathring{\mathfrak{g}}$. Let $\mathring{\Delta} \subset \mathring{\mathfrak{h}}^*$ be the set of roots of $\mathring{\mathfrak{g}}$. Choose a subset Δ_+ of positive roots, and let $\mathring{\Pi} = \{\alpha_1, \dots, \alpha_\ell\}$ be the set of simple roots, θ the highest root, θ_s the highest short root, $\Delta_- = -\Delta_+$, $\mathring{\rho} = \sum_{\alpha \in \mathring{\Delta}_+} \alpha/2$. Let $\mathring{Q} = \sum_{\alpha \in \mathring{\Delta}} \mathbb{Z}\alpha \subset \mathring{\mathfrak{h}}^*$, the root lattice of $\mathring{\mathfrak{g}}$. Set $\mathring{\mathfrak{n}} = \bigoplus_{\alpha \in \mathring{\Delta}_+} \mathring{\mathfrak{g}}_{\alpha}$, $\mathring{\mathfrak{n}}_- = \bigoplus_{\alpha \in \mathring{\Delta}_-} \mathring{\mathfrak{g}}_{\alpha}$, where $\mathring{\mathfrak{g}}_{\alpha}$ is the root space of $\mathring{\mathfrak{g}}$ with root α .

Let (\mid) be the normalized invariant bilinear form of $\mathring{\mathfrak{g}}$. We identify $\mathring{\mathfrak{h}}$ with $\mathring{\mathfrak{h}}^*$ using (\mid) . Let $\mathring{\Delta}^{\vee} = \{\alpha^{\vee} : \alpha \in \mathring{\Delta}\}$, the set of coroots, $\mathring{Q}^{\vee} = \sum_{\alpha \in \mathring{\Delta}} \mathbb{Z}\alpha^{\vee} \subset \mathring{\mathfrak{h}} = \mathring{\mathfrak{h}}^*$, the coroot lattice of $\mathring{\mathfrak{g}}$, where $\alpha^{\vee} = 2\alpha/(\alpha|\alpha)$.

Let $\overset{\circ}{\mathcal{W}} \subset Gl(\overset{\circ}{\mathfrak{h}}^*)$ be the Weyl group of $\overset{\circ}{\mathfrak{g}}$, $s_\alpha \in \overset{\circ}{\mathcal{W}}$ be the reflection corresponding to $\alpha \in \overset{\circ}{\Delta}$: $s_\alpha(\lambda) = \lambda - \lambda(\alpha^\vee)\alpha$.

Let \mathfrak{g} be the affine Kac-Moody algebra associated with $\overset{\circ}{\mathfrak{g}}$:

$$\mathfrak{g} = \overset{\circ}{\mathfrak{g}}[t, t^{-1}] \oplus \mathbb{C}K,$$

The commutation relations of \mathfrak{g} are given by

$$[xt^m, yt^n] = [x, y]t^{m+n} + m\delta_{m+n,0}(x|y)K, \quad [K, \mathfrak{g}] = 0.$$

We consider $\overset{\circ}{\mathfrak{g}}$ as a subalgebra of \mathfrak{g} by the natural embedding $\overset{\circ}{\mathfrak{g}} \hookrightarrow \mathfrak{g}$, $x \mapsto xt^0$. Let $\mathfrak{h} = \overset{\circ}{\mathfrak{h}} \oplus \mathbb{C}K$, the Cartan subalgebra of \mathfrak{g} , $\mathfrak{h}^* = \overset{\circ}{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0$ the dual of \mathfrak{g} . Here $\overset{\circ}{\mathfrak{h}}^*(K) = \Lambda_0(\overset{\circ}{\mathfrak{h}}) = 0$, $\Lambda_0(K) = 1$. The number $\langle \lambda, K \rangle$ is called the level of λ .

Let $\Delta_+^{\vee, re} = \overset{\circ}{\Delta}^\vee \sqcup \{\alpha^\vee + nK; \alpha \in \overset{\circ}{\Delta}_{long}, n \in \mathbb{N}\} \sqcup \{\alpha^\vee + r^\vee nK; \alpha \in \overset{\circ}{\Delta}_{short}, n \in \mathbb{Z}\}$ be the set of positive real coroots, $\Delta_-^{\vee, re} = -\Delta_+^{\vee, re}$ the set of negative real coroots, $\Delta^{\vee, re} = \Delta_+^{\vee, re} \sqcup \Delta_-^{\vee, re}$, the set of real roots, where $\overset{\circ}{\Delta}_{long}$ and $\overset{\circ}{\Delta}_{short}$ are the sets of long roots and short roots of $\overset{\circ}{\mathfrak{g}}$, respectively. Let $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_\ell^\vee, \alpha_0^\vee = -\theta + K\}$ the set of simple roots.

We have the following action $\alpha \mapsto t_\alpha$ of $\overset{\circ}{\mathfrak{h}}^*$ on \mathfrak{h}^* :

$$t_\alpha(\lambda) = \lambda + \langle \lambda, K \rangle \alpha, \quad \lambda \in \mathfrak{h}^*$$

Dually $\overset{\circ}{\mathfrak{h}}$ acts on \mathfrak{h} as $t_\alpha(h) = h - \langle \alpha, h \rangle K$. For a subset $L \subset \overset{\circ}{\mathfrak{h}}^*$ let $t_L = \{t_\alpha | \alpha \in L\}$. The Weyl group of \mathfrak{g} , or the affine Weyl group \mathcal{W} of $\overset{\circ}{\mathcal{W}}$, is the semidirect product

$$\mathcal{W} = \overset{\circ}{\mathcal{W}} \ltimes t_{\overset{\circ}{Q}^\vee}.$$

The extended Weyl group \mathcal{W}^e of the semidirect product

$$\mathcal{W}^e = \overset{\circ}{\mathcal{W}} \ltimes t_{\overset{\circ}{P}^\vee}$$

where $\overset{\circ}{P}^\vee = \{\lambda \in \overset{\circ}{\mathfrak{h}}; \langle \alpha, \lambda \rangle \in \mathbb{Z} \text{ for all } \alpha \in \overset{\circ}{\Delta}\}$, the coweight lattice of $\overset{\circ}{\mathfrak{g}}$. We have

$$\mathcal{W}^e = \mathcal{W}_+^e \ltimes \mathcal{W},$$

where \mathcal{W}_+^e subgroup of \mathcal{W}^e consisting of elements which fixes the set Π^\vee .

To define the set of roots of \mathfrak{g} , one needs to enlarge \mathfrak{h} by a one-dimensional space $\mathbb{C}D$ by letting $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}D = \overset{\circ}{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}D$ and extend the bilinear form $(\cdot | \cdot)$ from $\overset{\circ}{\mathfrak{h}}$ to $\tilde{\mathfrak{h}}$ by letting $(K|\tilde{\mathfrak{h}}) = (D|\tilde{\mathfrak{h}}) = (K|K) = (D|D) = 0$ and $(D|K) = 1$. We identify \mathfrak{h}^* with the subspace of $\tilde{\mathfrak{h}}^*$ consisting of elements which vanishes on D . Thus,

$$\tilde{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\delta = \overset{\circ}{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta,$$

where δ is defined by

$$\delta|_{\mathfrak{h}} = 0, \quad \langle \delta, D \rangle = 1.$$

The action of $\mathcal{W} \mathfrak{h}^*$ is extended to $\tilde{\mathfrak{h}}^*$ by

$$w(\delta) = \delta \quad w \in \mathring{\mathcal{W}},$$

$$t_\alpha(\lambda) = \lambda + \langle \Lambda, K \rangle \alpha - (\langle \lambda, \alpha \rangle + \frac{(\alpha|\alpha)}{2} \langle \lambda, K \rangle) \delta, \quad \lambda \in \mathfrak{h}^*.$$

For $\lambda \in \tilde{\mathfrak{h}}^*$ let $\bar{\lambda} \in \mathring{\mathfrak{h}}^*$, be its restriction to $\mathring{\mathfrak{h}}$.

Let $\Delta_+^{re} = \mathring{\Delta}_+ \sqcup \{\alpha + n\delta; \alpha \in \mathring{\Delta}, n \in \mathbb{N}\}$, the set of positive real roots, $\Delta_-^{re} = -\Delta_+^{re}$, $\Delta^{re} = \Delta_+^{re} \sqcup \Delta_-^{re}$ the set of real roots, $\Pi = \{\alpha_0 = -\theta + \delta, \alpha_1, \dots, \alpha_\ell\}$ the set of simple roots. The reflection s_α corresponding $\alpha = \bar{\alpha} + n\delta$ is given by $s_\alpha = t_{-n\bar{\alpha}^\vee} s_{\bar{\alpha}}$. We set $s_i = s_{\alpha_i}$ for $\alpha_i \in \Pi$.

3.2. Twisted Bruhat ordering. Let $\ell : \mathcal{W}^e \rightarrow \mathbb{Z}_{\geq 0}$ the length function: $\ell(w) = \# \Delta_+^{re} \cap w(\Delta_-^{re})$. We have

$$(6) \quad \ell(t_\mu y) = \sum_{\alpha \in \Delta_+ \cap y(\Delta_+)} |(\alpha|\mu)| + \sum_{\alpha \in \Delta_+ \cap y(\Delta_-)} |1 - (\alpha|\mu)|$$

for $\mu \in \mathring{P}^\vee$, $y \in \mathring{\mathcal{W}}$.

The *twisted length function* [Ark1] $\ell^y : \mathcal{W}^e \rightarrow \mathbb{Z}$ with the twist $y \in \mathcal{W}^e$ is defined by

$$\ell^y(w) = \# \Delta_+^{re} \cap w(\Delta_-^{re}) \cap y(\Delta_+^{re}) - \# \Delta_+^{re} \cap w(\Delta_-^{re}) \cap y(\Delta_-^{re}).$$

We often write $\ell^\alpha(w)$ for $\ell^{t_\alpha}(w)$.

Lemma 3.1. *Let $w, y \in \mathcal{W}^e$.*

(i) *Suppose that $\ell(ys_i) = \ell(y) + 1$ for $i \in I$. Then*

$$\ell^{ys_i}(w) = \begin{cases} \ell^y(w) & \text{if } w^{-1}y(\alpha_i) \in \Delta_+^{re}, \\ \ell^y(w) - 2 & \text{if } w^{-1}y(\alpha_i) \in \Delta_-^{re}. \end{cases}$$

(ii) $\ell^y(w) = \ell(y^{-1}w) - \ell(y^{-1})$.

Proof. (i) The assertion follows from the fact that $\Delta_+^{re} \cap ys_i(\Delta_-^{re}) = \Delta_+^{re} \cap y(\Delta_-^{re}) \sqcup \{y(s_i)\}$. (ii) We prove by induction on $\ell(y)$. If $\ell(y) = 0$ then $\ell^y(w) = \ell(w) = \ell(y^{-1}w)$. Suppose that $\ell(ys_i) = \ell(y) + 1$. If $w^{-1}y(\alpha_i) \in \Delta_+^{re}$ then $\ell(s_i y^{-1}w) = \ell(y^{-1}w) + 1$. Hence by (i) and induction hypothesis,

$$\ell^{ys_i}(w) = \ell^y(w) = \ell(y^{-1}w) - \ell(y^{-1}) = \ell(s_i y^{-1}w) - \ell(s_i y^{-1}).$$

If $w^{-1}y(\alpha_i) \in \Delta_-^{re}$ then $\ell(s_i y^{-1}w) = \ell(y^{-1}w) - 1$. Again by (i) and induction hypothesis,

$$\ell^{ys_i}(w) = \ell^y(w) - 2 = \ell(y^{-1}w) - 2 - \ell(y^{-1}) = \ell(s_i y^{-1}w) - \ell(s_i y^{-1}).$$

This completes the proof. \square

For $w_1, w_2, y \in \mathcal{W}$ and $\gamma \in \Delta^{re}$, write $w_1 \xrightarrow[y]{\gamma} w_2$ if $w_1 = s_\gamma w_2$ and $\ell^y(w_1) > \ell^y(w_2)$. Below, we shall often omit the symbol γ above the arrow. Also, we shall omit the symbol y under the arrow if $y = 1$. By Lemma 3.1 (ii) we have

$$(7) \quad w_1 \xrightarrow[y]{y(\gamma)} w_2 \iff y^{-1}w_1 \xrightarrow{y^{-1}\gamma} y^{-1}w_2.$$

In particular for $\beta = y(\alpha_i) \in \Delta_+^{re}$, $\alpha_i \in \Pi$, and $w_1, w_2 \in \mathcal{W}$ such that $\ell^y(w_2) - \ell^y(w_1) = 1$ we have the equivalence

$$(8) \quad \begin{array}{ccc} & s_\beta w_1 & \\ \nearrow y & & \nwarrow y \\ w_1 & & s_\beta w_2 \\ \searrow y & & \nearrow y \\ & w_2 & \end{array} \iff \begin{array}{ccc} & s_\beta w_1 & \\ \nwarrow y & & \nearrow y \\ & & s_\beta w_2 \\ \nearrow y & & \nwarrow y \\ & w_2 & \end{array}$$

by [BGG, Lemma 11.3]. Define $w \preceq_y w'$ if there exists a sequence w_1, w_2, \dots, w_k of elements of \mathcal{W} such that

$$w \xrightarrow{y} w_1 \xrightarrow{y} w_2 \xrightarrow{y} \dots \xrightarrow{y} w_k \xrightarrow{y} w'.$$

Note that

$$(9) \quad w \preceq_y w' \iff y^{-1}w' \preceq y^{-1}w,$$

by (7), where $\preceq = \preceq_1$, the (usual) Bruhat ordering of \mathcal{W} . It follows that \preceq_y defines a partial ordering of \mathcal{W} .

3.3. Semi-infinite Bruhat ordering. Define the *semi-infinite length* [FF2] $\ell^{\frac{\infty}{2}}(w)$ of $w \in \mathcal{W}^e$ by

$$\ell^{\frac{\infty}{2}}(w) = \#\{\alpha \in \Delta_+^{re} \cap w(\Delta_-^{re}), \bar{\alpha} \in \overset{\circ}{\Delta}_+\} - \#\{\alpha \in \Delta_+^{re} \cap w(\Delta_-^{re}), \bar{\alpha} \in \overset{\circ}{\Delta}_-\}.$$

We have

$$(10) \quad \ell^{\frac{\infty}{2}}(t_\lambda y) = \ell(y) - 2(\overset{\circ}{\rho}|\lambda)$$

for $\lambda \in \overset{\circ}{P}^\vee$, $w \in \overset{\circ}{\mathcal{W}}$.

Set

$$\overset{\circ}{P}_+^\vee = \{\lambda \in \overset{\circ}{P}^\vee; \alpha(\lambda) \geq 0 \text{ for all } \alpha \in \overset{\circ}{\Delta}_+\},$$

We say that $\lambda \in \overset{\circ}{P}_+^\vee$ is sufficiently large if $\alpha(\lambda)$ is sufficiently large for all $\alpha \in \overset{\circ}{\Delta}_+$.

By (6) and (10) we have the following assertion.

Lemma 3.2. $\ell^{\frac{\infty}{2}}(w) = -\ell^{-\lambda}(w)$ for a sufficiently large $\lambda \in \overset{\circ}{P}_+^\vee$.

We write

$$w_1 \xrightarrow[\frac{\infty}{2}]{\gamma} w_2$$

for $w_1, w_2 \in \mathcal{W}$ and $\gamma \in \Delta^{re}$ if $w_1 = w_2 s_\gamma$ and $\ell^{\frac{\infty}{2}}(w_1) < \ell^{\frac{\infty}{2}}(w_2)$. (We shall often omit the symbol γ above the arrow.) Define $w \preceq_{\frac{\infty}{2}} w'$ if there exists a sequence w_1, w_2, \dots, w_k of elements of \mathcal{W} such that

$$w \xrightarrow[\frac{\infty}{2}]{} w_1 \xrightarrow[\frac{\infty}{2}]{} w_2 \xrightarrow[\frac{\infty}{2}]{} \dots \xrightarrow[\frac{\infty}{2}]{} w_k \xrightarrow[\frac{\infty}{2}]{} w'.$$

By Lemma 3.2

$$w \preceq_{\frac{\infty}{2}} w' \iff w' \preceq_{t_\lambda} w \quad \text{for a sufficiently large } \lambda \in \overset{\circ}{P}_+^\vee.$$

It follows that $\preceq_{\frac{\infty}{2}}$ defines a partial ordering of \mathcal{W} . Following Arkhipov [Ark1], we call it the *semi-infinite Bruhat ordering* on \mathcal{W} . By [Soe1, Claim 4.14] the semi-infinite Bruhat ordering coincides with the ordering defined by Lusztig [Lus].

3.4. Semi-infinite analogue of parabolic subgroups. Let S be a subset of $\overset{\circ}{\Pi}$, $\overset{\circ}{\Delta}_S$ the subroot system of $\overset{\circ}{\Delta}$ generated by $\alpha_i \in S$, $\{\theta_1, \dots, \theta_s\}$ the set of longest roots of the simple root subsystems in $\overset{\circ}{\Delta}_S$.

Set

$$\Delta_S = \{\alpha + n\delta \in \Delta^{re}; \alpha \in \overset{\circ}{\Delta}_S, n \in \mathbb{Z}\}, \quad \mathcal{W}_S = \langle s_\alpha; \alpha \in \Delta_S \rangle \subset \mathcal{W}.$$

Then Δ_S is a subroot system of Δ^{re} isomorphic to the affine root system associated with $\overset{\circ}{\Delta}_S$. Put $\Delta_{S,+} = \Delta_S \cap \Delta_+^{re}$, the set of positive root of Δ_S . Then $\Pi_S = S \sqcup \{-\theta_1 + \delta, \dots, -\theta_s + \delta\}$ is a set of simple roots of Δ_S . We have $\mathcal{W}_S = \overset{\circ}{\mathcal{W}}_S \ltimes t_{\overset{\circ}{Q}_S^\vee}$, where $\overset{\circ}{Q}_S^\vee = \sum_{\alpha \in \overset{\circ}{\Delta}_S} \mathbb{Z}\alpha^\vee$. By (10), the restriction of the semi-infinite length function to \mathcal{W}_S coincides with the semi-infinite length function of the affine Weyl group \mathcal{W}_S . Define

$$\mathcal{W}^S = \{w \in \mathcal{W}; w^{-1}(\Delta_{S,+}) \subset \Delta_+^{re}\}.$$

Theorem 3.3. *The multiplication map $\mathcal{W}_S \times \mathcal{W}^S \rightarrow \mathcal{W}$, $(u, v) \mapsto uv$, is a bijection. Moreover, we have*

$$\ell^{\frac{\infty}{2}}(uv) = \ell^{\frac{\infty}{2}}(u) + \ell^{\frac{\infty}{2}}(v) \quad \text{for } u \in \mathcal{W}_S, v \in \mathcal{W}^S.$$

Proof. First, we show the injectivity of the multiplication map. Suppose that $u_1v_1 = u_2v_2$ with $u_i \in \mathcal{W}_S$, $v_i \in \mathcal{W}^S$. Then $v_1 = uv_2$ with $u = u_1^{-1}u_2 \in \mathcal{W}_S$. If $u \neq 1$ then there exists $\alpha \in \Delta_{S,+}$ such that $u^{-1}(\alpha) \in -\Delta_{S,+}$. But then $v_2 \in \mathcal{W}^S$ implies that $v_1^{-1}(\alpha) = v_2^{-1}u^{-1}(\alpha) \in \Delta_-^{re}$, and this contradicts that $v_1 \in \mathcal{W}^S$. Hence $u_1 = u_2$, and so $v_1 = v_2$.

Second, we show that the multiplication map $\mathcal{W}_S \times \mathcal{W}^S \rightarrow \mathcal{W}$ is surjective. We will prove by induction on $\sharp(w^{-1}(\Delta_{S,+}) \cap \Delta_-^{re})$ that there exists $u \in \mathcal{W}_S$ such that $u^{-1}w \in \mathcal{W}^S$. If $\sharp(w^{-1}(\Delta_{S,+}) \cap \Delta_-^{re}) = 0$, $w \in \mathcal{W}^S$ and there is nothing to show. Suppose that $\sharp(w^{-1}(\Delta_{S,+}) \cap \Delta_-^{re}) > 0$. Then there exists $\beta \in \Pi_S$ such that $w^{-1}(\beta) \in \Delta_-^{re}$. Indeed, any element $\alpha \in \Delta_{S,+}$ is expressed as $\alpha = \sum_{\beta \in \Pi_S} n_\beta \beta$ with $n_\beta \in \mathbb{Z}_{\geq 0}$. Thus $w^{-1}(\alpha) = \sum_{\beta \in \Pi_S} n_\beta w^{-1}(\beta) \in \Delta_-^{re}$ implies that one of $w^{-1}(\beta)$ must belong to Δ_-^{re} . Now because $(s_\beta w)^{-1}(\Delta_{S,+}) = w^{-1}s_\beta(\Delta_{S,+}) = w^{-1}(\Delta_{S,+} \setminus \{\beta\} \sqcup \{-\beta\}) = w^{-1}(\Delta_{S,+}) \setminus \{w^{-1}(\beta)\} \sqcup \{-w^{-1}(\beta)\}$,

$$(s_\beta w)^{-1}(\Delta_{S,+}) \cap \Delta_-^{re} = w^{-1}(\Delta_{S,+}) \cap \Delta_-^{re} \setminus \{w^{-1}(\beta)\}.$$

Hence by applying the induction hypothesis to $s_\beta w$ we find an element $u \in \mathcal{W}_S$ such that $u^{-1}s_\beta w \in \mathcal{W}^S$.

Finally, we prove the equality of the semi-infinite length. By (10), we have $\ell^{\frac{\infty}{2}}(t_\mu w) = \ell^{\frac{\infty}{2}}(t_\mu) + \ell^{\frac{\infty}{2}}(w)$ for any $\mu \in \overset{\circ}{Q}^\vee$. Hence we may assume that $u \in \overset{\circ}{\mathcal{W}}_S$. We will prove by induction on the length $\ell(u)$ of $u \in \overset{\circ}{\mathcal{W}}_S$ that $\ell^{\frac{\infty}{2}}(uv) = \ell^{\frac{\infty}{2}}(u) +$

$\ell^{\frac{\infty}{2}}(v)$ for any $v \in \mathcal{W}^S$. Suppose that $\ell(u) = 1$, so that $u = s_i$ for some $i \in S$. Let $v = t_\mu y \in \mathcal{W}^S$ with $\mu \in \mathring{Q}^\vee$, $y \in \mathring{\mathcal{W}}$. Note that $v \in \mathcal{W}^S$ is equivalent to that

$$(11) \quad \text{if } \alpha \in \mathring{\Delta}_{S,+} \text{ then } \alpha(\mu) = \begin{cases} 0 & \text{if } y^{-1}(\alpha) \in \mathring{\Delta}_+, \\ 1 & \text{if } y^{-1}(\alpha) \in \mathring{\Delta}_-. \end{cases}$$

Since

$$\ell^{\frac{\infty}{2}}(s_i t_\mu y) = \ell(t_{s_i(\mu)} s_i y) = \ell(s_i y) - 2(\rho|\mu - \alpha_i(\mu)\alpha_i^\vee) = \ell(s_i y) - 2(\rho|\mu) + 2\alpha_i(\mu),$$

(11) implies that $\ell^{\frac{\infty}{2}}(s_i v) = \ell^{\frac{\infty}{2}}(v) + 1$. Next let $u = s_i u_1 \in \mathring{\mathcal{W}}_S$ with $u_1 \in \mathring{\mathcal{W}}_S$, $i \in S$, $\ell(u) = \ell(u_1) + 1$, so that $u_1^{-1}(\alpha_i) \in \mathring{\Delta}_+$. Let $v = t_\mu y \in \mathcal{W}^S$ as above. We have

$$\ell^{\frac{\infty}{2}}(uv) = \ell^{\frac{\infty}{2}}(t_{s_i u_1(\mu)} s_i u_1 y) = \ell(s_i u_1 y) - 2(\rho|s_i u_1(\mu)).$$

If $\ell(s_i u_1 y) = \ell(u_1 y) + 1$, then $\mathring{\Delta}_+ \ni (u_1 y)^{-1}(\alpha_i) = y^{-1}(u_1^{-1}(\alpha_i))$. Hence $(\mu|u_1^{-1}(\alpha_i)) = 0$ by (11), which means $s_i u_1(\mu) = u_1(\mu)$. By the induction hypothesis, this proves that $\ell^{\frac{\infty}{2}}(uv) = \ell^{\frac{\infty}{2}}(u) + \ell^{\frac{\infty}{2}}(v)$. If $\ell(s_i u_1 y) = \ell(u_1 y) - 1$, then $\mathring{\Delta}_- \ni (u_1 y)^{-1}(\alpha_i) = y^{-1}(u_1^{-1}(\alpha_i))$. So (11) gives $(\mu|u_1^{-1}(\alpha_i)) = 1$, which means $s_i u_1(\mu) = u_1(\mu) - \alpha_i^\vee$. By the induction hypothesis, this proves that $\ell^{\frac{\infty}{2}}(uv) = \ell^{\frac{\infty}{2}}(u) + \ell^{\frac{\infty}{2}}(v)$ as required. \square

4. WAKIMOTO MODULES AND TWISTED VERMA MODULES

4.1. Wakimoto modules. Let $\mathfrak{g}, \mathfrak{h}$ be as in §3.1, and let us consider the \mathbb{Z} -grading of \mathfrak{g} with $\mathfrak{g}_0 = \mathfrak{h}$, $\mathfrak{g}_1 = \bigoplus_{\alpha \in \Pi} \mathfrak{g}_\alpha$, where \mathfrak{g}_α is the root space of \mathfrak{g} of root α . Let $\rho = \mathring{\rho} + h^\vee \Lambda_0 \in \mathfrak{h}^*$, where h^\vee is the dual Coxeter number of $\mathring{\mathfrak{g}}$. Then $\langle \rho, \alpha^\vee \rangle = 1$ for all $\alpha \in \Pi$ and 2ρ define a semi-infinite 1-cochain of \mathfrak{g} [Ark2].

Let $\mathcal{O}^\mathfrak{g}$ be the full subcategory of $\tilde{\mathcal{O}}^\mathfrak{g}$ consisting of modules on which \mathfrak{h} acts semisimply. For $M \in \mathcal{O}^\mathfrak{g}$ let $M_\mu = \{m \in M; hm = \mu(h)m \text{ for all } h \in \mathfrak{h}\}$, so that $M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$. The formal character of $M \in \mathcal{O}^\mathfrak{g}$ is defined by $\text{ch } M = \sum_{\mu \in \mathfrak{h}^*} (\dim_{\mathbb{C}} M_\mu) e^\mu$.

Let $L\mathring{\mathfrak{n}}, L\mathring{\mathfrak{n}}_-, \mathfrak{a}$ and $\bar{\mathfrak{a}}$ be graded subalgebras of \mathfrak{g} defined by

$$\begin{aligned} L\mathring{\mathfrak{n}} &= \mathring{\mathfrak{n}}[t, t^{-1}], & L\mathring{\mathfrak{n}}_- &= \mathring{\mathfrak{n}}_-[t, t^{-1}], \\ \mathfrak{a} &= L\mathring{\mathfrak{n}} \oplus \mathring{\mathfrak{h}}[t^{-1}]t^{-1}, & \bar{\mathfrak{a}} &= L\mathring{\mathfrak{n}}_- \oplus \mathring{\mathfrak{h}}[t] \oplus \mathbb{C}K. \end{aligned}$$

Then $0 = 2\rho|_{L\mathring{\mathfrak{n}}} = 2\rho|_{L\mathring{\mathfrak{n}}_-} = 2\rho|_{\mathfrak{a}}$ gives semi-infinite 1-cochains of $L\mathring{\mathfrak{n}}, L\mathring{\mathfrak{n}}_-, \mathfrak{a}$, and $2\rho|_{\bar{\mathfrak{a}}}$ gives a semi-infinite 1-cochain of $\bar{\mathfrak{a}}$.

Define the *Wakimoto module* $W(\lambda)$ of \mathfrak{g} with highest weight $\lambda \in \mathfrak{h}^*$ by

$$W(\lambda) = \text{S-ind}_{\bar{\mathfrak{a}}}^{\mathfrak{g}} \mathbb{C}_\lambda,$$

where \mathbb{C}_λ is the one-dimensional representation of \mathfrak{h} corresponding to λ regarded as a $\bar{\mathfrak{a}}$ -module by the natural projection $\bar{\mathfrak{a}} \rightarrow \mathfrak{h}$. By Lemma 2.5 we have

$$(12) \quad W(\lambda) \cong SS\mathfrak{a} \text{ as } \mathfrak{a}\text{-modules,}$$

and hence

$$(13) \quad H^{\frac{\infty}{2}+i}(\mathfrak{a}, W(\lambda)) \cong \begin{cases} \mathbb{C}_\lambda & \text{if } i = 0, \\ 0 & \text{otherwise} \end{cases} \text{ as } \mathfrak{h}\text{-modules},$$

$$(14) \quad \text{ch } W(\lambda) = \text{ch } M(\lambda),$$

and $W(\lambda)$ is an object of $\mathcal{O}^\mathfrak{g}$. Here $M(\lambda)$ is the Verma module of \mathfrak{g} with highest weight λ .

4.2. Uniqueness of Wakimoto modules.

Theorem 4.1. *Let $\lambda \in \mathfrak{h}^*$ be non-critical, that is, $\lambda(K) \neq -h^\vee$. Let M be an object of $\mathcal{O}^\mathfrak{g}$ such that*

$$H^{\frac{\infty}{2}+i}(\mathfrak{a}, M) \cong \begin{cases} \mathbb{C}_\lambda & \text{if } i = 0, \\ 0 & \text{otherwise} \end{cases}$$

as \mathfrak{h} -modules. Then M is isomorphic to $W(\lambda)$.

Let $\mathcal{H} = \mathring{\mathfrak{h}}[t, t^{-1}] \oplus \mathbb{C}K$, the Heisenberg subalgebra of \mathfrak{g} . Denote by π_λ the irreducible representation of \mathcal{H} with highest weight λ . Then for an object M of $\mathcal{O}^\mathfrak{g}$ of level k $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{n}}, M)$ is naturally an \mathcal{H} -module of level $k + h^\vee$.

Lemma 4.2. *Let λ, M be as in Theorem 4.6. Then the condition of Theorem 4.6 is equivalent to that*

$$H^{\frac{\infty}{2}+i}(L\mathring{\mathfrak{n}}, M) \cong \begin{cases} \pi_{\lambda+h^\vee\Lambda_0} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Note that the assumption that λ is non-critical implies that $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{n}}, M)$ is semi-simple as an \mathcal{H} -module and is a direct sum of π_μ s. Consider the Hochschild-Serre spectral sequence for the ideal $L\mathring{\mathfrak{n}} \subset \mathfrak{a}$. By definition, we have

$$E_2^{p,q} = \begin{cases} H_{-p}(\mathring{\mathfrak{h}}[t^{-1}]t^{-1}, H^{\frac{\infty}{2}+q}(L\mathring{\mathfrak{n}}, M)) & \text{for } p \leq 0, \\ 0 & \text{for } p > 0. \end{cases}$$

By the above mentioned fact $H^{\frac{\infty}{2}+q}(L\mathring{\mathfrak{n}}, M)$ is free over $\mathring{\mathfrak{h}}[t^{-1}]t^{-1}$. Hence

$$E_2^{p,q} = \begin{cases} H^{\frac{\infty}{2}+q}(L\mathring{\mathfrak{n}}, M) / \mathring{\mathfrak{h}}[t^{-1}]t^{-1}(H^{\frac{\infty}{2}+q}(L\mathring{\mathfrak{n}}, M)) & \text{for } p = 0. \\ 0 & \text{for } p \neq 0. \end{cases}$$

Therefore the spectral sequence collapses at $E_2 = E_\infty$, and $H^{\frac{\infty}{2}+i}(\mathfrak{a}, M) = 0$ for $i \neq 0$ if and only if $H^{\frac{\infty}{2}+i}(L\mathring{\mathfrak{n}}, M) = 0$ for $i \neq 0$. Also, $H^{\frac{\infty}{2}+0}(\mathfrak{a}, M) \cong \mathbb{C}_\lambda$ is equivalent to $H^{\frac{\infty}{2}+0}(L\mathring{\mathfrak{n}}, M) \cong \pi_{\lambda+h^\vee\Lambda_0}$. This completes the proof. \square

Lemma 4.3. *Let M be as in Theorem 4.6. Then $M \cong SS\mathfrak{a}$ as \mathfrak{a} -modules.*

Proof. By Proposition 2.3 it is sufficient to show that $\text{S-ind}_\mathfrak{a}^\mathfrak{a} M \cong SS\mathfrak{a}$. So we compute $H^{\frac{\infty}{2}+\bullet}(\mathfrak{a}, SS\mathfrak{a} \otimes M)$. For this, we consider the Hochschild-Serre spectral sequence for the ideal $L\mathring{\mathfrak{n}} \subset \mathfrak{a}$ as in the proof of Lemma 4.2. By definition we have

$$(15) \quad E_1^{\bullet,q} = H^{\frac{\infty}{2}+q}(L\mathring{\mathfrak{n}}, SS\mathfrak{a} \otimes_\mathbb{C} M) \otimes_\mathbb{C} \bigwedge^\bullet(\mathring{\mathfrak{h}}[t^{-1}]t^{-1}),$$

$$(16) \quad E_2^{p,q} = H_{-p}(\mathring{\mathfrak{h}}[t^{-1}]t^{-1}, H^{\frac{\infty}{2}+q}(L\mathring{\mathfrak{n}}, SS\mathfrak{a} \otimes_\mathbb{C} M)).$$

Set

$$F^p SS\mathfrak{a} = \bigoplus_{\langle \mu, \check{\rho} \rangle \geq p} SS\mathfrak{a}_\mu,$$

$$F^p(M \otimes_{\mathbb{C}} \bigwedge^{\frac{\infty}{2} + \bullet}(L\check{\mathfrak{n}})) = U(\check{\mathfrak{h}}[t^{-1}]t^{-1}) \otimes_{\mathbb{C}} F^p SS\mathfrak{a} \otimes_{\mathbb{C}} M \otimes_{\mathbb{C}} \bigwedge^{\frac{\infty}{2} + \bullet}(L\check{\mathfrak{n}}),$$

where $SS\mathfrak{a}_\mu = \{s \in SS\mathfrak{a}; hs - sh = \mu(h)s \text{ for } h \in \check{\mathfrak{h}}\}$. It is easy to see that $\{F^p(SS\mathfrak{a} \otimes_{\mathbb{C}} M \otimes_{\mathbb{C}} \bigwedge^{\frac{\infty}{2} + \bullet}(L\check{\mathfrak{n}}))\}$ defines a decreasing, weight-wise regular filtration of the complex $SS\mathfrak{a} \otimes_{\mathbb{C}} M \otimes_{\mathbb{C}} \bigwedge^{\frac{\infty}{2} + \bullet}(L\check{\mathfrak{n}})$. Consider the associated spectral sequence $E'_r \Rightarrow H^{\frac{\infty}{2} + \bullet}(L\check{\mathfrak{n}}, SS\mathfrak{a} \otimes_{\mathbb{C}} M)$. Because the associated graded space $\text{gr } SS\mathfrak{a}$ with respect to this filtration is a trivial $L\check{\mathfrak{n}}$ -module the E_1 -term of the spectral sequence is isomorphic to $SS\mathfrak{a} \otimes_{\mathbb{C}} H^{\frac{\infty}{2} + \bullet}(L\check{\mathfrak{n}}, M)$. Hence by the hypothesis and Lemma 4.2 the spectral sequence E'_r collapses at $E'_1 = E'_\infty$ and we obtain the isomorphism of $\check{\mathfrak{h}}$ -modules

$$(17) \quad \text{gr}^F H^{\frac{\infty}{2} + i}(L\check{\mathfrak{n}}, SS\mathfrak{a} \otimes_{\mathbb{C}} M) \cong \begin{cases} SS\mathfrak{a} \otimes_{\mathbb{C}} U(\check{\mathfrak{h}}[t^{-1}]t^{-1}) & (i = 0), \\ 0 & (i \neq 0). \end{cases}$$

By the weight consideration one finds that the composition

$$H^{\frac{\infty}{2} + 0}(L\check{\mathfrak{n}}, SS\mathfrak{a} \otimes_{\mathbb{C}} M) \xrightarrow{\sim} \text{gr } H^{\frac{\infty}{2} + 0}(L\check{\mathfrak{n}}, SS\mathfrak{a} \otimes_{\mathbb{C}} M) \xrightarrow{\sim} SS\mathfrak{a} \otimes_{\mathbb{C}} U(\check{\mathfrak{h}}[t^{-1}]t^{-1})$$

of the symbol map with the isomorphism (17) is a homomorphism of left \mathfrak{a} -modules. Thus by (15) we have

$$E_1^{\bullet, q} \cong \begin{cases} SS\mathfrak{a} \otimes_{\mathbb{C}} U(\check{\mathfrak{h}}[t^{-1}]t^{-1}) \otimes_{\mathbb{C}} \bigwedge^{\bullet}(\check{\mathfrak{h}}[t^{-1}]t^{-1}) & (q = 0), \\ 0 & (q \neq 0) \end{cases}$$

as left \mathfrak{a} -modules. It follows that the E_2 -term of the Hochschild-Serre spectral sequence $E_r \Rightarrow H^{\frac{\infty}{2} + \bullet}(\mathfrak{a}, SS\mathfrak{g} \otimes_{\mathbb{C}} M)$ is given by

$$(18) \quad E_2^{p, q} \cong \begin{cases} SS\mathfrak{a} & (p = q = 0), \\ 0 & (\text{otherwise.}) \end{cases}$$

as \mathfrak{a} -modules. Therefore the spectral sequence collapses at $E_2 = E_\infty$ and we obtain $M \cong SS\mathfrak{a}$ as required. \square

Proof of Theorem 4.6. By Proposition 2.3, it is sufficient to show that $\text{S-ind}_{\mathfrak{g}}^{\mathfrak{g}}(M) \cong W(\lambda)$. Let (C^\bullet, d) denote the complex $SS\mathfrak{g} \otimes_{\mathbb{C}} M \otimes_{\mathbb{C}} \bigwedge^{\frac{\infty}{2} + \bullet}(\mathfrak{g})$ that defines the semi-infinite cohomology $H^{\frac{\infty}{2} + \bullet}(\mathfrak{g}, SS\mathfrak{g} \otimes_{\mathbb{C}} M)$. Note that we have the decomposition $\bigwedge^{\frac{\infty}{2} + \bullet}(\mathfrak{g}) = \bigwedge^{\frac{\infty}{2} + \bullet}(\bar{\mathfrak{a}}) \otimes_{\mathbb{C}} \bigwedge^{\frac{\infty}{2} + \bullet}(\mathfrak{a})$. Set

$$C^{p, q} = SS\mathfrak{g} \otimes_{\mathbb{C}} M \otimes_{\mathbb{C}} \bigwedge^{\frac{\infty}{2} + p}(\bar{\mathfrak{a}}) \otimes_{\mathbb{C}} \bigwedge^{\frac{\infty}{2} + q}(\mathfrak{a}),$$

so that $C^n = \bigoplus_{p+q=n} C^{p, q}$. One sees that the differential d decomposes as $d = d_{\mathfrak{a}} + d_{\bar{\mathfrak{a}}}$, where $d_{\mathfrak{a}} C^{p, q} \subset C^{p, q+1}$, $d_{\bar{\mathfrak{a}}} C^{p, q} \subset C^{p+1, q}$. Since $d^2 = 1$, we have $d_{\mathfrak{a}}^2 = d_{\bar{\mathfrak{a}}}^2 = \{d_{\mathfrak{a}}, d_{\bar{\mathfrak{a}}}\} = 0$. In particular there is a spectral sequence (E_r, d_r) such that $d_0 = d_{\mathfrak{a}}$ and $d_1 = d_{\bar{\mathfrak{a}}}$.

The E_1 -term of this spectral sequence is $H^{\frac{\infty}{2} + \bullet}(\mathfrak{a}, U^{\frac{\infty}{2}}(\mathfrak{g}) \otimes_{\mathbb{C}} M \otimes_{\mathbb{C}} \bigwedge^{\frac{\infty}{2} + \bullet}(\bar{\mathfrak{a}}))$, where $U^{\frac{\infty}{2}}(\mathfrak{g}) \otimes_{\mathbb{C}} M \otimes_{\mathbb{C}} \bigwedge^{\frac{\infty}{2} + \bullet}(\bar{\mathfrak{a}})$ is regarded as a \mathfrak{a} -module by the action $x(u \otimes m \otimes \omega) =$

$-ux \otimes m \otimes \omega + u \otimes xm \otimes \omega + u \otimes m \otimes \omega$ and $\bigwedge^{\frac{\infty}{2}+\bullet}(\bar{\mathfrak{a}})$ is regarded as a \mathfrak{a} -module by the identification $\bar{\mathfrak{a}} = \mathfrak{g}/\mathfrak{a}$. Therefore by Proposition 2.3 and Lemma 4.3 we have

$$(19) \quad E_1^{\bullet,q} \cong \begin{cases} U^{\frac{\infty}{2}}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathbb{C}_{\lambda} \otimes_{\mathbb{C}} \bigwedge^{\frac{\infty}{2}+\bullet}(\bar{\mathfrak{a}}) & (i = 0) \\ 0 & (i \neq 0) \end{cases}$$

of left \mathfrak{g} -modules. One finds under the isomorphism (19) the complex $(E_1^{\bullet,0}, d_{\bar{\mathfrak{a}}})$ can be identified with the complex for calculating $H^{\frac{\infty}{2}+\bullet}(\bar{\mathfrak{a}}, SS\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}_{\lambda})$. Therefore we obtain the isomorphism

$$E_2^{p,q} \cong \begin{cases} W(\lambda) & (p = q = 0), \\ 0 & (\text{otherwise}) \end{cases}$$

as \mathfrak{g} -modules. Hence the spectral sequence collapses at $E_2 = E_{\infty}$, and we obtain the desired isomorphism. \square

Remark 4.4. For a critical $\lambda \in \mathfrak{h}^*$ one can show in the same manner as Theorem 4.6 that the *restricted Wakimoto module* [FF3] $W^{res}(\lambda)$ with highest weight λ is characterized by the property

$$H^{\frac{\infty}{2}+i}(L\mathfrak{n}, W^{res}(\lambda)) \cong \begin{cases} \mathbb{C}_{\lambda} & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.5. *Suppose that $\langle \lambda + \rho, K \rangle \notin \mathbb{Q}_{\geq 0}$. Then $W(t_{\alpha} \circ \lambda) \cong M(t_{\alpha} \circ \lambda)$ for a sufficiently large $\alpha \in P_+^{\vee}$.*

Proof. Let α be sufficiently large. By the hypothesis $\langle t_{\alpha}(\lambda + \rho), \beta^{\vee} \rangle \notin \mathbb{N}$ for all $\beta \in \Delta_+^{re}$ such that $\bar{\beta} \in \mathring{\Delta}_+$. Therefore it follows from [A1, Theorem 3.1] that $M(t_{\alpha} \circ \lambda)$ is cofree over $\mathring{\mathfrak{n}}[t] = \mathfrak{a}_+$. Because $M(t_{\alpha} \circ \lambda)$ is obviously free over \mathfrak{a}_- we have $H^{\frac{\infty}{2}+i}(\mathfrak{a}, M(t_{\alpha} \circ \lambda)) \cong \begin{cases} \mathbb{C}_{t_{\alpha} \circ \lambda} & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases}$ Thus the assertion follows from Theorem 4.6. \square

4.3. Twisted Wakimoto modules. By abuse of notation we denote also by w a Tits lifting of $w \in \mathcal{W}^e$ to $\text{Aut}(\mathfrak{g})$.

For $w \in \mathring{\mathcal{W}}$ we have the decomposition $\mathfrak{g} = w(\mathfrak{a}) \oplus w(\bar{\mathfrak{a}})$ and 2ρ defines a semi-infinite 1-cochain of the graded subalgebra $w(\bar{\mathfrak{a}})$. Hence we can define the Wakimoto module $W^w(\lambda)$ with highest weight λ and twist $w \in \mathring{\mathcal{W}}$ by

$$W^w(\lambda) = \text{S-ind}_{w(\bar{\mathfrak{a}})}^{\mathfrak{g}} \mathbb{C}_{\lambda},$$

where \mathbb{C}_{λ} is the one-dimensional representation of \mathfrak{h} corresponding to λ naturally regarded as a $\bar{\mathfrak{a}}$ -module. We have

$$\begin{aligned} W^w(\lambda) &\cong SS w(\mathfrak{a}) \text{ as } w(\mathfrak{a})\text{-modules and } \text{ch } W^w(\lambda) = \text{ch } M(\lambda), \\ H^{\frac{\infty}{2}+i}(w(\mathfrak{a}), W^w(\lambda)) &\cong \begin{cases} \mathbb{C}_{\lambda} & (i = 0), \\ 0 & \text{otherwise} \end{cases} \text{ as } \mathfrak{h}\text{-modules.} \end{aligned}$$

The following assertion can be proved in the same manner as Theorem 4.6.

Theorem 4.6. *Let $\lambda \in \mathfrak{h}^*$ be non-critical, $w \in \mathring{\mathcal{W}}$. Let M be an object of $\mathcal{O}^{\mathfrak{g}}$ such that*

$$H^{\frac{\infty}{2}+i}(w(\mathfrak{a}), M) \cong \begin{cases} \mathbb{C}_{\lambda} & \text{if } i = 0, \\ 0 & \text{otherwise} \end{cases}$$

as \mathfrak{h} -modules. Then M is isomorphic to $W^w(\lambda)$.

4.4. Twisted Verma modules. Let $\mathfrak{b} = \bigoplus_{n \geq 0} \mathfrak{g}_n$, the Borel subalgebra of \mathfrak{g} , $\mathfrak{n}_- = \bigoplus_{n < 0} \mathfrak{g}_n$, so that $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{b}$.

For $w \in \mathcal{W}^e$ and $\lambda \in \mathfrak{h}^*$, define the twisted Verma module $M^w(\lambda)$ with highest weight λ by

$$M^w(\lambda) = \text{S-ind}_{w(\mathfrak{b})}^{\mathfrak{g}} \mathbb{C}_{\lambda},$$

where \mathbb{C}_{λ} is the one-dimensional representation of \mathfrak{h} defined by λ regarded as a $w(\mathfrak{b})$ -module by the natural projection $w(\mathfrak{b}) \rightarrow \mathfrak{h}$. We have $M^w(\lambda) \cong SS w(\mathfrak{n}_-)$ as $w(\mathfrak{n}_-)$ -modules by Lemma 2.5 and hence

$$(20) \quad H^{\frac{\infty}{2}+i}(w(\mathfrak{n}_-), M^w(\lambda)) \cong \begin{cases} \mathbb{C}_{\lambda} & (i = 0), \\ 0 & (i \neq 0) \end{cases} \text{ as } \mathfrak{h}\text{-modules},$$

$$(21) \quad \text{ch } M^w(\lambda) = \text{ch } M(\lambda).$$

In particular $M^w(\lambda)$ is an object of $\mathcal{O}^{\mathfrak{g}}$.

The following assertion can be shown in the same manner as Theorem 4.6.

Theorem 4.7. *Let M be an object $\mathcal{O}^{\mathfrak{g}}$ such that*

$$H^{\frac{\infty}{2}+i}(w(\mathfrak{n}_-), M) \cong \begin{cases} \mathbb{C}_{\lambda} & \text{for } i = 0, \\ 0 & \text{for } i \neq 0 \end{cases}$$

as \mathfrak{h} -modules. Then M is isomorphic to $M^w(\lambda)$.

We write $M^{\alpha}(\lambda)$ for $M^{t_{\alpha}}(\lambda)$, $\alpha \in \mathring{P}^{\vee}$.

4.5. Wakimoto modules as inductive limits of twisted Verma modules.

Let $\alpha \in \mathring{P}_+^{\vee}$. Then $t_{-\alpha}(\mathfrak{b})_- \subset \bar{\mathfrak{a}}_-$ and $\bar{\mathfrak{a}}_+ \subset t_{-\alpha}(\mathfrak{b})_+$. By [Ark1, Lemma 6.1.7] the natural map

$$\bigwedge^{\frac{\infty}{2}+\bullet}(t_{-\alpha}(\mathfrak{b})) \rightarrow \bigwedge^{\frac{\infty}{2}+\bullet}(t_{-\alpha}(\mathfrak{b}) \cap \bar{\mathfrak{a}}) \rightarrow \bigwedge^{\frac{\infty}{2}+\bullet}(\bar{\mathfrak{a}})$$

induces the homomorphism

$$(22) \quad \phi_{\alpha}^{\lambda} : M^{-\alpha}(\lambda) \rightarrow W(\lambda)$$

of \mathfrak{g} -modules. If $\alpha_1, \alpha_2 \in \mathring{P}_+^{\vee}$ such that $\alpha_2 - \alpha_1 \in \mathring{P}_+^{\vee}$ then $t_{-\alpha_1}(\mathfrak{b})_- \subset t_{-\alpha_2}(\mathfrak{b})_-$ and $t_{-\alpha_2}(\mathfrak{b})_+ \subset t_{-\alpha_1}(\mathfrak{b})_+$. Hence by [Ark1, Lemma 6.1.7] the natural map

$$\bigwedge^{\frac{\infty}{2}+\bullet}(t_{-\alpha_1}(\mathfrak{b})) \rightarrow \bigwedge^{\frac{\infty}{2}+\bullet}(t_{-\alpha_1}(\mathfrak{b}) \cap t_{-\alpha_2}(\mathfrak{b})) \rightarrow \bigwedge^{\frac{\infty}{2}+\bullet}(t_{-\alpha_2}(\mathfrak{b}))$$

induces the homomorphism

$$(23) \quad \phi_{\alpha_2, \alpha_1}^{\lambda} : M^{-\alpha_1}(\lambda) \rightarrow M^{-\alpha_2}(\lambda)$$

of \mathfrak{g} -modules. By construction we have

$$\phi_{\alpha_2}^{\lambda} \circ \phi_{\alpha_2, \alpha_1}^{\lambda} = \phi_{\alpha_1}^{\lambda}.$$

Let $\{\alpha_1, \alpha_2, \dots\}$ be a sequence in $\overset{\circ}{P}_+^\vee$ such that $\alpha_i - \alpha_{i-1} \in \overset{\circ}{P}_+^\vee$ and $\lim_{n \rightarrow \infty} \beta(\alpha_n) = \infty$ for all $\beta \in \overset{\circ}{\Delta}_+$. Then $\{M^{-\alpha_n}(\lambda) : \phi_{\alpha_n, \alpha_n}^\lambda\}$ forms an inductive system of \mathfrak{g} -modules. By [Ark1, Lemma 6.4.2] we have the isomorphism

$$(24) \quad W(\lambda) \cong \varinjlim_n M^{-\alpha_n}(\lambda)$$

of \mathfrak{g} -modules. Note that

$$(25) \quad W(\lambda)_\mu \cong M^{-\alpha_n}(\lambda)_\mu \quad \text{for a sufficiently large } n$$

for each $\mu \in \mathfrak{h}^*$.

The same argument show that

$$(26) \quad W^y(\lambda) \cong \varinjlim_n M^{-y(\alpha_n)}(\lambda)$$

for $y \in \overset{\circ}{W}$, where $\{\alpha_1, \alpha_2, \dots\}$ is a sequence as above.

5. TWISTING FUNCTORS

5.1. Twisting functors. (See [Ark1, Ark3, AL, AS, A1] for the details.) For each $w \in \mathcal{W}^e$ a twisting functor $T_w : \mathcal{O}^\mathfrak{g} \rightarrow \mathcal{O}^\mathfrak{g}$ is defined as follows. Let $\mathfrak{n}_w = \mathfrak{n}_- \cap w^{-1}(\mathfrak{n}_+)$ and set $N_w = U(\mathfrak{n}_w)$. Define

$$S_w = U \otimes_{N_w} N_w^*.$$

It is known that S_w has a U -bimodule structure and $S_w \cong N_w^* \otimes_{N_w} U$ as right U -modules and left N_w -modules. For $M \in \mathcal{O}^\mathfrak{g}$ define

$$T_w(M) = \phi_w(S_w \otimes_{U(\mathfrak{g})} M),$$

where ϕ_w means that the action of \mathfrak{g} is twisted by the automorphism w of \mathfrak{g} . This define a right exact functor $T_w : \mathcal{O}^\mathfrak{g} \rightarrow \mathcal{O}^\mathfrak{g}$ such that

$$(27) \quad T_{ws_i} \cong T_w T_i \quad \text{if } \alpha_i \in \Pi, \ell(ws_i) = \ell(w) + 1,$$

where $T_i = T_{s_i}$.

The functor T_w admits a right adjoint functor G_w in the category $\mathcal{O}^\mathfrak{g}$:

$$G_w(M) = \text{Hom}_U(S_w, \phi_w^{-1}(M)).$$

Lemma 5.1. *Let $M \in \mathcal{O}^\mathfrak{g}$, $w \in \mathcal{W}^e$*

- (i) *Suppose that M is free over \mathfrak{n}_w . Then $M \cong G_w T_w(M)$.*
- (ii) *Suppose that M is cofree over $w(\mathfrak{n}_w)$. Then $M \cong T_w G_w(M)$.*

Proposition 5.2 ([Ark1, Lemma 6.3.1]). *For $\lambda \in \mathfrak{h}^*$ and $w \in \mathcal{W}^e$ we have*

$$M^w(\lambda) \cong T_w M(w^{-1} \circ \lambda).$$

By Proposition 5.2 and [AL, Proposition 6.3], it follows that

$$(28) \quad M^w(\lambda) \cong M(\lambda) \quad \text{if } \langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{N} \quad \text{for all } \alpha \in \Delta_+^{re} \cap w(\Delta_-^{re}).$$

Proposition 5.3. *For $w \in \mathcal{W}^e$ the functor T_w gives the isomorphism*

$$\text{Hom}_\mathfrak{g}(M(\lambda), M(\mu)) \xrightarrow{\sim} \text{Hom}_\mathfrak{g}(M^w(w \circ \lambda), M^w(w \circ \mu))$$

for $\lambda, \mu \in \mathfrak{h}^*$. The inverse map is given by G_w .

Proof. By Lemma 5.1 (1) and Proposition 5.2 we have

$$\mathrm{Hom}_{\mathfrak{g}}(M(\lambda), M(\mu)) \cong \mathrm{Hom}_{\mathfrak{g}}(M(\lambda), G_w T_w M(\mu)) \cong \mathrm{Hom}_{\mathfrak{g}}(T_w M(\lambda), T_w M(\mu)).$$

□

We set $T_\alpha = T_{t_\alpha}$, $G_\alpha = G_{t_\alpha}$ for $\alpha \in \mathring{P}^\vee$.

Proposition 5.4. *Let $\alpha \in \mathring{P}_+^\vee$, $\lambda \in \mathfrak{h}^*$.*

- (i) $T_{-\alpha} W(\lambda) \cong W(t_{-\alpha} \circ \lambda)$.
- (ii) $G_{-\alpha} W(\lambda) \cong W(t_\alpha \circ \lambda)$.

Proof. (i) Let $\{\gamma_1, \gamma_2, \dots\}$ be a sequence in \mathring{P}_+^\vee such that $\gamma_i - \gamma_{i-1} \in \mathring{P}_+^\vee$ and $\lim_{n \rightarrow \infty} \beta(\gamma_n) = \infty$ for all $\beta \in \mathring{\Delta}_+$. Set $\gamma'_i = \gamma_i + \alpha$. Then the sequence $\{\gamma'_1, \gamma'_2, \dots\}$ satisfies the same property. Hence by (24), (27), Proposition 5.2 and the fact that a homology functor commutes with inductive limits we have $T_{-\alpha} W(\lambda) \cong T_{-\alpha}(\varinjlim M^{-\gamma_i}(\lambda)) = \varinjlim T_{-\alpha} M^{-\gamma_i}(\lambda) = \varinjlim T_{-\alpha} T_{-\gamma_i} M(-t_{\gamma_i} \circ \lambda) = \varinjlim T_{-\gamma'_i} M(-t_{\gamma_i} \circ \lambda) = \varinjlim M^{-\gamma'_i}(t_\alpha \circ \lambda) \cong W(t_\alpha \circ \lambda)$. (ii) Since $\mathfrak{n}_{t_{-\alpha}} \subset \mathfrak{a}_-$, $W(\lambda)$ is free over $\mathfrak{n}_{t_{-\alpha}}$. Hence $W(t_\alpha \circ \lambda) = G_{-\alpha} T_{-\alpha} W(t_\alpha \circ \lambda) \cong G_{-\alpha} W(\lambda)$ by Lemma 5.1 and (i). □

Corollary 5.5. *For $\alpha \in \mathring{P}_+^\vee$, the functor $G_{-\alpha}$ gives the isomorphism*

$$\mathrm{Hom}_{\mathfrak{g}}(W(\lambda), W(\mu)) \cong \mathrm{Hom}_{\mathfrak{g}}(W(t_\alpha \circ \lambda), W(t_\alpha \circ \mu)).$$

for $\lambda, \mu \in \mathfrak{h}^$. The inverse map is given by $T_{-\alpha}$.*

The following assertion follows from Proposition 4.5 and Corollary 5.5.

Proposition 5.6. *Let $\lambda, \mu \in \mathfrak{h}^*$ be of level k , and suppose that $k + h^\vee \notin \mathbb{Q}_{\geq 0}$. Then*

$$\mathrm{Hom}_{\mathfrak{g}}(W(\lambda), W(\mu)) \cong \mathrm{Hom}_{\mathfrak{g}}(M(t_\alpha \circ \lambda), M(t_\alpha \circ \mu))$$

for a sufficiently large $\alpha \in \mathring{P}_+^\vee$.

5.2. Left derived functors of twisting functors. The functor T_w , $w \in \mathcal{W}^e$, admits the left derived functor in the category $\mathcal{O}^{\mathfrak{g}}$ since it is a Lie algebra homology functor:

$$\mathcal{L}_i T_w(M) = \phi_w(H_i(\mathfrak{g}, S_w \otimes_{\mathbb{C}} M)),$$

where \mathfrak{g} acts on $N_w^* \otimes_{\mathbb{C}} M$ by the action $X(f \otimes m) = -fX \otimes m + f \otimes Xm$. Note that

$$(29) \quad \mathcal{L}_i T_w(M) \cong \phi_w(H_i(\mathfrak{n}_w, N_w^* \otimes_{\mathbb{C}} M))$$

as $w(\mathfrak{n}_w)$ -modules. In particular we have the following.

Lemma 5.7. *Suppose $M \in \mathcal{O}^{\mathfrak{g}}$ is free over \mathfrak{n}_w . Then $\mathcal{L}_i T_w(M) = 0$ for $i \geq 1$.*

Set $I := \{0, \dots, \ell\}$ and let $\{e_i, h_i, f_i; i \in I\}$, $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$, be the Chevalley generators of \mathfrak{g} . For $i \in I$, let $\mathfrak{sl}_2^{(i)}$ denote the copy of \mathfrak{sl}_2 in \mathfrak{g} spanned by $\{e_i, h_i, f_i\}$

Proposition 5.8. *Let $M \in \mathcal{O}^{\mathfrak{g}}$, $i \in I$. Denote by N the largest $\mathfrak{sl}_2^{(i)}$ -integrable submodule of M . Then $T_i(M) \cong T_i(M/N)$, $\mathrm{ch} \mathcal{L}_1 T_i(M) \cong \mathrm{ch} N$ and $\mathcal{L}_p T_i(M) = 0$ for $p \geq 2$.*

Proof. Let $T_i^{(i)}$ denote the twisting functor for $\mathfrak{sl}_2^{(i)}$ corresponding to the reflection s_{α_i} . Because $T_i(M) \cong T_i^{(i)}(M)$ as $\mathfrak{sl}_2^{(i)}$ -modules and \mathfrak{h} -modules, we have

$$(30) \quad \mathcal{L}_i T_i(M) \cong \mathcal{L}_i T_a^{(i)}(M) \quad \text{as } \mathfrak{sl}_2^{(i)}\text{-modules and } \mathfrak{h}\text{-modules.}$$

In particular $\mathcal{L}_1 T_a(M) = 0$ for $i \geq 2$. Thus the exact sequence

$$0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0.$$

yields the long exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{L}_1 T_i(N) \rightarrow \mathcal{L}_1 T_i(M) \rightarrow \mathcal{L}_1 T_i(M/N) \\ \rightarrow T_i(N) \rightarrow T_i(M) \rightarrow T_i(M/N) \rightarrow 0. \end{aligned}$$

Because M/N is free as $\mathbb{C}[f_i]$ -module $\mathcal{L}_1 T_i(M/N) = 0$ by (30). Also, $T_i(N) = 0$ and $\mathcal{L}_1 T_i(N) \cong N$ as \mathfrak{h} -modules by [AS, Theorem 6.1] and (30). This completes the proof. \square

Let $L(\lambda) \in \mathcal{O}^{\mathfrak{g}}$ be the irreducible representation of \mathfrak{g} with highest weight λ .

Theorem 5.9 ([AS, Theorem 6.1]). *Let $\lambda \in \mathfrak{h}^*$ and suppose that $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$ with $\alpha_i \in \Pi$. Then*

$$\mathcal{L}_p T_i(L(\lambda)) \cong \begin{cases} L(\lambda) & \text{if } p = 1, \\ 0 & \text{if } p \neq 1. \end{cases}$$

Proof. The hypothesis implies that $L(\lambda)$ is $\mathfrak{sl}_2^{(i)}$ -integrable. Therefore $\mathcal{L}_p T_i(L(\lambda)) = 0$ for $p \neq 1$ and $\text{ch } \mathcal{L}_1 T_i(L(\lambda)) = \text{ch } L(\lambda)$ by Proposition 5.8. \square

5.3. Twisting functors associated with integral Weyl group. For $\lambda \in \mathfrak{h}^*$ let $\Delta(\lambda)$ and $\mathcal{W}(\lambda)$ be its integral root system and integral Weyl group, respectively:

$$\begin{aligned} \Delta(\lambda) &= \{\alpha \in \Delta^{re}; \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}\}, \\ \mathcal{W}(\lambda) &= \langle s_\alpha; \alpha \in \Delta(\lambda) \rangle \subset \mathcal{W}. \end{aligned}$$

Let $\Delta(\lambda)_+ = \Delta(\lambda) \cap \Delta_+^{re}$ the set of positive roots of $\Delta(\lambda)$, $\Pi(\lambda) \subset \Delta(\lambda)_+$ the set of simple roots of $\Delta(\lambda)$, $\ell_\lambda : \mathcal{W}(\lambda) \rightarrow \mathbb{Z}_{\geq 0}$ the length function.

Lemma 5.10. *Let $\lambda \in \mathfrak{h}^*$, $\alpha \in \Pi(\lambda)$. There exists $x \in \mathcal{W}$ and $\alpha_i \in \Pi$ such that $\alpha = x(\alpha_i)$ and $\ell(s_\alpha) = 2\ell(x) + 1$. We have $\Delta(\lambda)_+ \cap x(\Delta_-^{re}) = \emptyset$, $\Delta(x^{-1} \circ \lambda) = x^{-1}(\Delta(\lambda))$ and*

$$\mathcal{W}(x^{-1} \circ \lambda) = x^{-1} \mathcal{W}(\lambda) x \cong \mathcal{W}(\lambda).$$

Proof. The first assertion can be proved by induction on $\ell(s_\alpha)$. The second assertion follows from the first assertion and the hypothesis that $\alpha \in \Pi(\lambda)$. The remaining two assertions follow from the second. \square

Let $\mathcal{O}_{[\lambda]}^{\mathfrak{g}}$ be the block of $\mathcal{O}^{\mathfrak{g}}$ corresponding to λ , that is, the full subcategory of $\mathcal{O}^{\mathfrak{g}}$ consisting of objects M such that $[M : L(\mu)] \neq 0 \Rightarrow \mu \in \mathcal{W}(\lambda) \circ \mu$, where $[M : L(\mu)]$ is the multiplicity of $L(\mu)$ in the local composition factor of M .

Lemma 5.11. *Let $\lambda \in \mathfrak{h}^*$, $\alpha_i \in \Pi$. Suppose that $\langle \lambda + \rho, \alpha_i^\vee \rangle \notin \mathbb{Z}$. Then $T_i M(\lambda) \cong M(s_i \circ \lambda)$, $T_i L(w \circ \lambda) \cong L(s_i w \circ \lambda)$ for $w \in \mathcal{W}(\lambda)$. Moreover T_i gives an equivalence of categories $\mathcal{O}_{[\lambda]}^{\mathfrak{g}} \xrightarrow{\sim} \mathcal{O}_{[s_i \circ \lambda]}^{\mathfrak{g}}$.*

Proof. The first assertion follows from (28). By [A1, Theorems 3.1, 3.2] any object of $\mathcal{O}_{[\lambda]}^{\mathfrak{g}}$ and $\mathcal{O}_{[s_i \circ \lambda]}^{\mathfrak{g}}$ is free over $\mathbb{C}[f_i]$ and cofree over $\mathbb{C}[e_i]$. Hence by Lemma 5.1 T_i gives the equivalence of categories $\mathcal{O}_{[\lambda]}^{\mathfrak{g}} \xrightarrow{\sim} \mathcal{O}_{[s_i \circ \lambda]}^{\mathfrak{g}}$ with a quasi-inverse G_i . It follows that $T_i L(\lambda)$ is a simple \mathfrak{g} -module which is a quotient of $T_i M(\lambda) = M(s_i \circ \lambda)$, and hence is isomorphic to $L(s_i \circ \lambda)$. \square

Lemma 5.12. *Let λ, α_i be as in Lemma 5.11. Then $T_i^2 : \mathcal{O}_{[\lambda]}^{\mathfrak{g}} \rightarrow \mathcal{O}_{[\lambda]}^{\mathfrak{g}}$ is isomorphic to the identity functor.*

Proof. By Lemma 5.11 T_i^2 induces an auto-equivalence of the category $\mathcal{O}_{[\lambda]}^{\mathfrak{g}}$ such that $T_i^2 M(w \circ \lambda) \cong M(w \circ \lambda)$ and $T_i^2(L(w \circ \lambda)) \cong L(w \circ \lambda)$ for all $w \in \mathcal{W}(\lambda)$. The standard argument shows that such a functor must be an identify functor. \square

The following assertion also follows from (27) and Lemma 5.12.

Proposition 5.13. *Let $\lambda \in \mathfrak{h}^*$, $w = s_\alpha y \in \mathcal{W}(\lambda)$, $y \in \mathcal{W}(\lambda)$, $\alpha \in \Pi(\lambda)$, $\ell_\lambda(w) = \ell_\lambda(y) + 1$. Then $T_w : \mathcal{O}_{[\lambda]}^{\mathfrak{g}} \rightarrow \mathcal{O}_{[\lambda]}^{\mathfrak{g}}$ is isomorphic to the functor $T_{s_\alpha} \circ T_y : \mathcal{O}_{[\lambda]}^{\mathfrak{g}} \rightarrow \mathcal{O}_{[\lambda]}^{\mathfrak{g}}$.*

Lemma 5.14. *Let λ, α_i be as in Lemma 5.11. Then $T_i M^w(\lambda) \cong M^{s_i w s_i}(s_i \circ \lambda)$ for $w \in \mathcal{W}(\lambda)$.*

Proof. We have $T_i M^w(\lambda) \cong T_i T_w M(w^{-1} \circ \lambda) \cong T_i T_w T_i M(s_i w^{-1} \circ \lambda)$ by Lemma 5.11. Hence the assertion follows from (27) and Lemma 5.12. \square

Lemma 5.15. *Let $\lambda \in \mathfrak{h}^*$, $w \in \mathcal{W}(\lambda)$, $\alpha \in \Pi(\lambda)$ such that $\ell_\lambda(s_\alpha w) = \ell_\lambda(w) + 1$. Then $G_{s_\alpha} M^{s_\alpha w}(\lambda) \cong M^w(s_\alpha \circ \lambda)$.*

Proof. Let $x \in \mathcal{W}$ and $\alpha_i \in \Pi$ be as in Lemma 5.10. Then $T_{s_\alpha} = T_x T_i T_{x^{-1}}$, $G_\alpha = G_{x^{-1}} G_i G_x$. By Lemmas 5.10, 5.12, 5.14 and Proposition 5.13,

$$\begin{aligned} G_{s_\alpha} M^{s_\alpha w}(\lambda) &\cong G_{x^{-1}} G_i G_x T_x T_i T_{x^{-1}} M^w(s_\alpha w \circ \lambda) \cong G_{x^{-1}} G_i T_i T_{x^{-1}} M^w(s_\alpha w \circ \lambda) \\ &\cong G_{x^{-1}} G_i T_i M^{x^{-1}wx}(x^{-1}s_\alpha w \circ \lambda). \end{aligned}$$

We have $s_i, x^{-1}wx \in \mathcal{W}(x^{-1}\lambda)$ and $\ell_{x^{-1}\circ\lambda}(s_i x^{-1}wx) = \ell_{x^{-1}\circ\lambda}(x^{-1}wx) + 1$, and hence, $\ell(s_i x^{-1}wx) = \ell(x^{-1}wx) + 1$. Therefore

$$G_i T_i M^{x^{-1}wx}(x^{-1}s_\alpha w \circ \lambda) \cong M^{x^{-1}wx}(x^{-1}s_\alpha w \circ \lambda).$$

Hence we conclude that $G_{s_\alpha} M^{s_\alpha w}(\lambda) \cong G_{x^{-1}} M^{x^{-1}wx}(x^{-1}s_\alpha w \circ \lambda) \cong G_{x^{-1}} T_x M^w(s_\alpha w \circ \lambda) \cong M^w(s_\alpha w \circ \lambda)$ by Lemmas 5.12 and 5.14 as required. \square

Recall that a weight $\lambda \in \mathfrak{h}^*$ is called *regular dominant* if $\langle \lambda + \rho, \alpha^\vee \rangle \notin \{0, -1, -2, \dots\}$ for all Δ_+^{re} .

Theorem 5.16. *Let λ be regular dominant, $w, w', y \in \mathcal{W}(\lambda)$. Then*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathfrak{g}}(M^y(w \circ \lambda), M^y(w' \circ \lambda)) = \begin{cases} 1 & \text{if } w \preceq_y w', \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Lemma 5.15 we have

$$\operatorname{Hom}_{\mathfrak{g}}(M^y(w \circ \lambda), M^y(w' \circ \lambda)) \cong \operatorname{Hom}_{\mathfrak{g}}(M(y^{-1}w \circ \lambda), M(y^{-1}w' \circ \lambda)).$$

Hence the assertion follows from (9) and [KT, Proposition 2.5.5 (ii)]. \square

Proposition 5.17. *Let $\lambda \in \mathfrak{h}^*$, $w \in \mathcal{W}(\lambda)$, $\alpha \in \Pi(\lambda)$ and suppose that $\langle w(\lambda + \rho), \alpha^\vee \rangle \notin \mathbb{N}$. Then the following sequence is exact:*

$$0 \rightarrow M(s_\alpha w \circ \lambda) \xrightarrow{\varphi_1} M(w \circ \lambda) \xrightarrow{\varphi_2} M^{s_\alpha}(w \circ \lambda) \xrightarrow{\varphi_3} M^{s_\alpha}(s_\alpha w \circ \lambda) \rightarrow 0,$$

where $\varphi_1, \varphi_2, \varphi_3$ are any non-trivial \mathfrak{g} -homomorphisms.

Proof. First note that $\text{Hom}_{\mathfrak{g}}(M(s_\alpha w \circ \lambda), M(w \circ \lambda))$, $\text{Hom}_{\mathfrak{g}}(M(w \circ \lambda), M^{s_\alpha}(w \circ \lambda))$ and $\text{Hom}_{\mathfrak{g}}(M^{s_\alpha}(w \circ \lambda), M(s_\alpha w \circ \lambda))$ are all one-dimensional.

By Lemma 5.10 there exists $x \in \mathcal{W}$ and $\alpha_i \in \Pi$ such that $\alpha = x(\alpha_i)$ and $\ell(s_\alpha) = 2\ell(x) + 1$. Since $x^{-1}(\Delta(\lambda)_+) \subset \Delta_+^{re}$ by (28) we have

$$M(y \circ \lambda) \cong T_x M(x^{-1}y \circ \lambda),$$

$$M^{s_\alpha}(y \circ \lambda) = T_x T_i T_{x^{-1}} M(x s_i x^{-1} y \circ \lambda) \cong T_x T_i M(s_i x^{-1} y \circ \lambda) \cong T_x M^{s_i} M(x^{-1} y \circ \lambda)$$

for $y \in \mathcal{W}(\lambda)$.

Now since $\langle x^{-1}w(\lambda + \rho), \alpha_i^\vee \rangle = \langle w(\lambda + \rho), \alpha^\vee \rangle \in \mathbb{N}$ there is an exact sequence

$$0 \rightarrow M(s_i x^{-1} w \circ \lambda) \rightarrow M(x^{-1} w \circ \lambda) \rightarrow M^{s_i}(x^{-1} w \circ \lambda) \rightarrow M^{s_i}(s_i x^{-1} w \circ \lambda) \rightarrow 0$$

by [AL, Propostion 6.2]. Applying the exact functor $T_x : \mathcal{O}_{[x^{-1} \circ \lambda]}^{\mathfrak{g}} \rightarrow \mathcal{O}_{[\lambda]}^{\mathfrak{g}}$, we obtain the required exact sequence. \square

Proposition 5.18. *Let $\lambda \in \mathfrak{h}^*$, $\alpha \in \Pi(\lambda)$, $M \in \mathcal{O}_{[\lambda]}^{\mathfrak{g}}$. Take $\alpha_i \in \Pi$, $x \in \mathcal{W}$ such that $\alpha = x(\alpha_i)$ and $\ell(s_\alpha) = 2\ell(x) + 1$ as in Lemma 5.10. Let N' be the largest $\mathfrak{sl}_2^{(i)}$ -integrable submodule of $T_{x^{-1}}(M)$ and set $N = T_x(N') \subset M$. Then $T_\alpha(M) \cong T_{s_\alpha}(M/N)$, $\text{ch } \mathcal{L}_1 T_{s_\alpha}(M) = \text{ch } N$ and $\mathcal{L}_p T_{s_\alpha}(M) = 0$ for $p \geq 2$.*

Proof. We have $T_\alpha = T_x T_i T_{x^{-1}}$, and $T_{x^{-1}} : \mathcal{O}_{[\lambda]}^{\mathfrak{g}} \rightarrow \mathcal{O}_{[x^{-1} \circ \lambda]}^{\mathfrak{g}}$, $T_x : \mathcal{O}_{[x^{-1} \circ \lambda]}^{\mathfrak{g}} \rightarrow \mathcal{O}_{[\lambda]}^{\mathfrak{g}}$ are exact functors. Therefore $\mathcal{L}_p T_{s_\alpha}(M) = T_x(\mathcal{L}_p T_i(T_{x^{-1}} M))$. It follows from Proposition 5.8 that $T_{s_\alpha}(M) = T_x T_i T_{x^{-1}}(M) \cong T_x T_i(T_{x^{-1}} M/N') \cong T_x T_i T_{x^{-1}}(M/N) = T_{s_\alpha}(M/N)$, $\text{ch } \mathcal{L}_1 T_{s_\alpha}(M) = \text{ch } T_x T_{x^{-1}}(N) = \text{ch } N$, and $\mathcal{L}_p T_{s_\alpha}(M) = 0$ for $p \geq 2$. \square

Theorem 5.19. *Let $\lambda \in \mathfrak{h}^*$ be regular dominant weight, $w \in \mathcal{W}(\lambda)$. Then*

$$\mathcal{L}_i T_w(L(\lambda)) \cong \begin{cases} L(\lambda) & \text{if } i = \ell_\lambda(w), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\alpha \in \Pi(\lambda)$. From Theorem 5.9 and Lemma 5.12 it follows that

$$(31) \quad \mathcal{L}_i T_{s_\alpha}(L(\lambda)) \cong \begin{cases} L(\lambda) & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 5.13 the assertion follows in the same as in [AS, Corollary 6.2]. \square

6. TWO-SIDED BGG RESOLUTIONS OF ADMISSIBLE REPRESENTATIONS

6.1. Admissible representations. A weight $\lambda \in \mathfrak{h}^*$ is called *admissible* if it is regular dominant and

$$\mathbb{Q}\Delta(\lambda) = \mathbb{Q}\Delta^{re}.$$

The irreducible representation $L(\lambda)$ is called admissible if λ is admissible. A complex number k is called an *admissible number* for \mathfrak{g} if the weight $k\Lambda_0$ is admissible.

Let r^\vee be the lacing number of $\overset{\circ}{\mathfrak{g}}$, that is, the maximal number of the edges of the Dynkin digram of $\overset{\circ}{\mathfrak{g}}$. Also, let h be the Coxeter number of $\overset{\circ}{\mathfrak{g}}$.

Proposition 6.1 ([KW2, KW3]). *A complex number k is admissible if and only if*

$$(32) \quad k + h^\vee = \frac{p}{q} \quad \text{with } p, q \in \mathbb{N}, (p, q) = 1, p \geq \begin{cases} h^\vee & \text{if } (r^\vee, q) = 1, \\ h & \text{if } (r^\vee, q) = r^\vee. \end{cases}$$

In this case $\Pi(k\Lambda_0) = \{\alpha_1, \dots, \alpha_\ell, \dot{\alpha}_0\}$, where $\dot{\alpha}_0 = \begin{cases} -\theta + q\delta & \text{if } (r^\vee, q) = 1, \\ -\theta_s + \frac{q}{r^\vee}\delta & \text{if } (r^\vee, q) = r^\vee. \end{cases}$

An admissible number k of the form (32) is called an admissible number with denominator q . For an admissible number k with denominator q we have

$$\mathcal{W}(k\Lambda_0) = \dot{\mathcal{W}} \rtimes t_{\dot{q}\dot{Q}^\vee} \cong \mathcal{W} \quad \text{if } (r^\vee, q) = 1,$$

$$\mathcal{W}(k\Lambda_0) = \dot{\mathcal{W}} \rtimes t_{\dot{q}\dot{Q}} \cong {}^L\mathcal{W} \quad \text{if } (r^\vee, q) = r^\vee.$$

where ${}^L\mathcal{W}$ is the Weyl group of the non-twisted affine Kac-Moody algebra ${}^L\mathfrak{g}$ associated with the Langlands dual ${}^L\dot{\mathfrak{g}}$ of $\dot{\mathfrak{g}}$. We set $\dot{s}_0 = s_{\dot{\alpha}_0} \in \mathcal{W}(k\Lambda_0)$, so that $\mathcal{W}(k\Lambda_0) = \langle s_1, \dots, s_\ell, \dot{s}_0 \rangle$.

For an admissible number k let Pr_k^+ be the set of admissible weights λ of level k such that $\lambda(\alpha^\vee) \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \dot{\Delta}_+$. Then $\{L(\lambda); \lambda \in Pr_k^+\}$ is the set of irreducible admissible representations of level k which are integrable over $\dot{\mathfrak{g}}$. We have $\Delta(\lambda) = \Delta(k\Lambda_0)$ for $\lambda \in Pr_k^+$. Next, for an admissible number k , let Pr_k be the set of admissible weights λ of level k such that $\Delta(\lambda) \cong \Delta(k\Lambda_0)$. Then [KW2]

$$(33) \quad Pr_k = \bigcup_{\substack{y \in \mathcal{W}^e \\ y(\Delta(k\Lambda_0)) \subset \Delta_+^{re}}} Pr_{k,y}, \quad Pr_{k,y} = y \circ Pr_k^+.$$

Note that

$$(34) \quad \mathcal{W}(\lambda) = y\mathcal{W}(k\Lambda_0)y^{-1} \quad \text{for } \lambda \in Pr_{k,y}.$$

For $\lambda \in Pr_k$, let $\ell_\lambda^y(?)$ with $y \in \mathcal{W}(\lambda)$ and $\ell_\lambda^{\frac{\infty}{2}}(?)$ be the twisted length function and the semi-infinite length function of the affine Weyl group $\mathcal{W}(\lambda)$, respectively. Also, the twisted Bruhat ordering \preceq_y , and the semi-infinite Bruhat ordering $\preceq_{\frac{\infty}{2}}$ are defined for $\mathcal{W}(\lambda)$.

The admissible weight $\lambda \in Pr_k$ is called the *principal admissible weight* if $\Delta(\lambda) \cong \Delta^{re}$, that is, if the denominator q of k is prime to r^\vee .

6.2. Feibig's equivalence and BGG resolution of admissible representations. The following theorem is the special case of a result of Feibig [Fie, Theorem 11].

Theorem 6.2 ([Fie]). *Let λ be regular dominant. Suppose that there exists a symmetrizable Kac-Moody algebra \mathfrak{g}' whose Weyl group \mathcal{W}' is isomorphic to $\mathcal{W}(\lambda)$. Let λ' be an integral dominant weight of \mathfrak{g}' , $\mathcal{O}_{[\lambda']}^{\mathfrak{g}'}$ the block of $\mathcal{O}^{\mathfrak{g}'}$ containing the irreducible highest weight representation $L^{\mathfrak{g}'}(\lambda')$ of \mathfrak{g}' with highest weight λ' . Then there is an equivalence of categories*

$$\mathcal{O}_{[\lambda]}^{\mathfrak{g}} \cong \mathcal{O}_{[\lambda']}^{\mathfrak{g}'}$$

which maps $M(w \circ \lambda)$ and $L(w \circ \lambda)$, $w \in \mathcal{W}(\lambda)$, to $M^{\mathfrak{g}'}(\phi(w) \circ \lambda')$ and $L^{\mathfrak{g}'}(\phi(w) \circ \lambda')$, respectively. Here $M^{\mathfrak{g}'}(\lambda')$ is the Verma module of \mathfrak{g}' with highest weight λ' and $\phi : \mathcal{W}(\lambda) \xrightarrow{\sim} \mathcal{W}'$ is the isomorphism.

Let k be an admissible number with denominator q , $\lambda \in Pr_k$. By Theorem 6.2 the block $\mathcal{O}_{[\lambda]}^{\mathfrak{g}}$ is equivalent to a block of the category \mathcal{O} of \mathfrak{g} or ${}^L\mathfrak{g}$ containing an integrable representation. In particular the existence of a BGG resolution of an integrable representation of an affine Kac-Moody algebra [GL, RCW] implies the existence of a BGG resolution for $L(\lambda)$:

Theorem 6.3. *Let k be an admissible number, $\lambda \in Pr_k$. Then there exists a complex*

$$\mathcal{B}_{\bullet}(\lambda) : \cdots \xrightarrow{d_3} \mathcal{B}_2(\lambda) \xrightarrow{d_2} \mathcal{B}_1(\lambda) \xrightarrow{d_1} \mathcal{B}_0(\lambda) \xrightarrow{d_0} 0$$

of the form $\mathcal{B}_i(\lambda) = \bigoplus_{\substack{w \in \mathcal{W}(\lambda) \\ \ell_{\lambda}(w) = i}} M(w \circ \lambda)$, $d_i = \sum_{\substack{w, w' \in \mathcal{W}(\lambda) \\ w \rightarrow w'}} d_{w', w}$, $d_{w, w'} \in \text{Hom}_{\mathfrak{g}}(M(w \circ \lambda), M(w' \circ \lambda))$, such that

$$H_i(\mathcal{B}_{\bullet}(\lambda)) \cong \begin{cases} L(\lambda) & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The resolution of $L(\lambda)$ in Theorem 6.3 can be combinatorially constructed as follows [BGG]: Fix a \mathfrak{g} -homomorphisms $i_{w', w}^{\lambda} : M(w \circ \lambda) \rightarrow M(w' \circ \lambda)$ for $w, w' \in \mathcal{W}(\lambda)$ with $w \preceq w'$ such that $i_{w'', w'}^{\lambda} \circ i_{w', w}^{\lambda} = i_{w'', w}^{\lambda}$ if $w \preceq w' \preceq w''$.

Theorem 6.4. *Let k be an admissible number, $\lambda \in Pr_k$. Assign $\epsilon_{w_1, w_2} \in \mathbb{C}^*$ for every arrow $w_1 \rightarrow w_2$ of $\mathcal{W}(\lambda)$ in such a way that $\epsilon_{w_1, w_2} \epsilon_{w_2, w_4} + \epsilon_{w_1, w_3} \epsilon_{w_3, w_4} = 0$ for every square (w_1, w_2, w_3, w_4) of $\mathcal{W}(\lambda)$ (such an assignment is possible by [BGG]). Set $d_{w, w'} = \epsilon_{w', w} i_{w', w}^{\lambda}$, $d_i = \sum_{\substack{w, w' \in \mathcal{W}(\lambda) \\ \ell_{\lambda}(w) = i, w \rightarrow w'}} d_{w, w'}$. Then*

$$\mathcal{B}_{\bullet}(\lambda) : \cdots \xrightarrow{d_3} \mathcal{B}_2(\lambda) \xrightarrow{d_2} \mathcal{B}_1(\lambda) \xrightarrow{d_1} \mathcal{B}_0(\lambda) \xrightarrow{d_0} 0,$$

where $\mathcal{B}_i(\lambda) = \bigoplus_{\substack{w \in \mathcal{W}(\lambda) \\ \ell(w) = i}} M(w \circ \lambda)$, is a resolution of $L(\lambda)$.

6.3. Twisted BGG resolution. For $w_1, w_2, y \in \mathcal{W}(\lambda)$ with $w_1 \preceq_y w_2$, let

$$\varphi_{w_2, w_1}^{\lambda, y} = T_y(i_{y^{-1}w_2, y^{-1}w_1}^{\lambda}) : M^y(w_1 \circ \lambda) \rightarrow M^y(w_2 \circ \lambda),$$

see Proposition 5.3.

Theorem 6.5. *Let k be an admissible number, $\lambda \in Pr_k$, $y \in \mathcal{W}(\lambda)$. Assign $\epsilon_{w_1, w_2}^y \in \mathbb{C}^*$ for every y -twisted arrow $w_1 \rightarrow w_2$ of $\mathcal{W}(\lambda)$ in such a way that $\epsilon_{w_1, w_2}^y \epsilon_{w_2, w_4}^y + \epsilon_{w_1, w_3}^y \epsilon_{w_3, w_4}^y = 0$ for every y -twisted square (w_1, w_2, w_3, w_4) of $\mathcal{W}(\lambda)$. Set $\mathcal{B}_i^y(\lambda) = \bigoplus_{\substack{w \in \mathcal{W}(\lambda) \\ \ell_{\lambda}^y(w) = i}} M^y(w \circ \lambda)$, $d_{w, w'}^y = \epsilon_{w', w}^y \varphi_{w', w}^{\lambda, y}$, $d_i = \sum_{\substack{w, w' \in \mathcal{W}(\lambda) \\ \ell_{\lambda}^y(w) = i, w \xrightarrow{y} w'}} d_{w, w'} : \mathcal{B}_i^y(\lambda) \rightarrow \mathcal{B}_{i-1}^y(\lambda)$. Then*

$$\mathcal{B}_{\bullet}^y(\lambda) : \cdots \xrightarrow{d_3} \mathcal{B}_2^y(\lambda) \xrightarrow{d_2} \mathcal{B}_1^y(\lambda) \xrightarrow{d_1} \mathcal{B}_0^y(\lambda) \xrightarrow{d_0} \mathcal{B}_1^y(\lambda) \rightarrow \cdots \rightarrow \mathcal{B}_{-\ell(y)}^y(\lambda) \rightarrow 0$$

is a complex of \mathfrak{g} -modules such that

$$H_i(B_\bullet^y(\lambda)) \cong \begin{cases} L(\lambda) & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Set $\epsilon_{y^{-1}w_1, y^{-1}w_2} = \epsilon_{w_1, w_2}^y$. Then $\{\epsilon_{w_1, w_2}^y\}$ satisfies the condition in Theorem if and only if $\{\epsilon_{y^{-1}w_1, y^{-1}w_2}\}$ satisfies the condition in Theorem 6.3. In particular such an assignment is possible. Consider the BGG resolution $\mathcal{B}_\bullet(\lambda)$ of $L(\lambda)$ in Theorem 6.3 associated with this assignment. We have $\mathcal{B}_\bullet^y(\lambda) = T_y(\mathcal{B}_\bullet(\lambda))[-\ell(y)]$, where $[-\ell(y)]$ denotes the shift of the degree. Therefore the assertion follows from Proposition 5.2, Lemma 5.7 and Theorem 5.19. \square

6.4. System of twisted BGG resolutions.

Proposition 6.6. *Let $\lambda \in \mathfrak{h}^*$ be regular dominant, $y = s_{\beta_1}s_{\beta_2}\dots s_{\beta_l}$ a reduced expression of $y \in \mathcal{W}(\lambda)$ with $\beta_i \in \Pi(\lambda)$. Set $y_i = s_{\beta_1}s_{\beta_2}\dots s_{\beta_i}$ for $i = 0, 1, \dots, l$ and fix a non-zero \mathfrak{g} -homomorphism $\phi_w^{y_i} : M^{y_i}(w \circ \lambda) \rightarrow M^{y_{i+1}}(w \circ \lambda)$ for $w \in \mathcal{W}(\lambda)$, $i = 1, \dots, l$. Then one can assign $\epsilon_{w_1, w_2}^{y_i} \in \mathbb{C}^*$ for each y_i -twisted arrow $w_1 \xrightarrow{y_i} w_2$ in such a way that the following hold for all $i = 1, \dots, l$.*

- (i) $\epsilon_{w_1, w_2}^{y_i} \epsilon_{w_2, w_4}^{y_i} + \epsilon_{w_1, w_3}^{y_i} \epsilon_{w_3, w_4}^{y_i} = 0$ for every y_i -twisted square (w_1, w_2, w_3, w_4) of $\mathcal{W}(\lambda)$,
- (ii) If $w_1 \xrightarrow{y_i} w_2$, $w_1 \xrightarrow{y_{i-1}} w_2$, $\ell^{y_i}(w_1) = \ell^{y_{i-1}}(w_1)$ and $\ell^{y_i}(w_2) = \ell^{y_{i-1}}(w_2)$, then the following diagram commutes.

$$(35) \quad \begin{array}{ccc} M^{y_{i-1}}(w_1 \circ \lambda) & \xrightarrow{\epsilon_{w_1, w_2}^{y_{i-1}} \varphi_{w_2, w_1}^{\lambda, y_{i-1}}} & M^{y_{i-1}}(w_2 \circ \lambda) \\ \phi_{w_1}^{y_{i-1}} \downarrow & & \downarrow \phi_{w_2}^{y_{i-1}} \\ M^y(w_1 \circ \lambda) & \xrightarrow{\epsilon_{w_1, w_2}^y \varphi_{w_2, w_1}^{\lambda, y}} & M^y(w_2 \circ \lambda). \end{array}$$

Proposition 6.7. *Let $\lambda \in \mathfrak{h}^*$ be regular dominant, $y \in \mathcal{W}(\lambda)$, $\alpha \in \Pi(\lambda)$ such that $\ell_\lambda(y s_\alpha) = \ell_\lambda(y) + 1$. Set $\beta = y(\alpha)$*

- (i) *Let $w_1, w_2 \in \mathcal{W}(\lambda)$. Suppose that $w_1 \xrightarrow{y} w_2$, $w_1 \xrightarrow{y s_\alpha} w_2$ and $\ell_\lambda^y(w_1) = \ell_\lambda^{y s_\alpha}(w_1)$. Then*

$$\dim_{\mathbb{C}} \text{Hom}_{\mathfrak{g}}(M^y(w_1 \circ \lambda), M^{y s_\alpha}(w_2 \circ \lambda)) = 1.$$

Moreover, both of the followings span $\text{Hom}_{\mathfrak{g}}(M^y(w_1 \circ \lambda), M^{y s_\alpha}(w_2 \circ \lambda))$.

- (a) *the composition $M^y(w_1 \circ \lambda) \rightarrow M^y(w_2 \circ \lambda) \rightarrow M^{y s_\alpha}(w_2 \circ \lambda)$ of any non-trivial \mathfrak{g} -homomorphisms,*
- (b) *the composition $M^y(w_1 \circ \lambda) \rightarrow M^{y s_\alpha}(w_1 \circ \lambda) \rightarrow M^{y s_\alpha}(w_2 \circ \lambda)$ of any non-trivial \mathfrak{g} -homomorphisms.*
- (ii) *Let $w_1, w_2 \in \mathcal{W}(\lambda)$. Suppose that $\ell_\lambda^y(w_1) = \ell_\lambda^y(w_2) + 2$ and $w_i^{-1}(\beta) \in \Delta_+^{r_e}$ for $i = 1, 2$. Then the composition $M^y(w_1 \circ \lambda) \rightarrow M^y(w_2 \circ \lambda) \rightarrow M^{y s_\alpha}(w_2 \circ \lambda)$ of any non-trivial homomorphisms is non-zero.*
- (iii) *Let $w \in \mathcal{W}(\lambda)$ and suppose that $s_\alpha w \xrightarrow{y} w$. Then the composition $M^y(s_\alpha w \circ \lambda) \rightarrow M^y(w \circ \lambda) \rightarrow M^{y s_\alpha}(w \circ \lambda)$ of any \mathfrak{g} -homomorphisms is zero.*

Proof. (i) First note that

$$\begin{aligned} \operatorname{Hom}_{\mathfrak{g}}(M^y(w_1 \circ \lambda), M^y(w_2 \circ \lambda)) &\cong \operatorname{Hom}_{\mathfrak{g}}(M(y^{-1}w_1 \circ \lambda), M(y^{-1}w_2 \circ \lambda)), \\ \operatorname{Hom}_{\mathfrak{g}}(M^y(w_1 \circ \lambda), M^{ys_\alpha}(w_1 \circ \lambda)) &\cong \operatorname{Hom}_{\mathfrak{g}}(M(y^{-1}w_1 \circ \lambda), M^{s_\alpha}(y^{-1}w_1 \circ \lambda)), \\ \operatorname{Hom}_{\mathfrak{g}}(M^y(w_2 \circ \lambda), M^{ys_\alpha}(w_2 \circ \lambda)) &\cong \operatorname{Hom}_{\mathfrak{g}}(M(y^{-1}w_2 \circ \lambda), M^{s_\alpha}(y^{-1}w_2 \circ \lambda)), \\ \operatorname{Hom}_{\mathfrak{g}}(M^{ys_\alpha}(w_1 \circ \lambda), M^{ys_\alpha}(w_2 \circ \lambda)) &\cong \operatorname{Hom}_{\mathfrak{g}}(M(s_\alpha y^{-1}w_1 \circ \lambda), M(s_\alpha y^{-1}w_2 \circ \lambda)) \end{aligned}$$

by Lemma 5.18. In particular they are all one-dimensional. Also, the same lemma implies that

$$\operatorname{Hom}(M^y(w_1 \circ \lambda), M^{ys_\alpha}(w_2 \circ \lambda)) \cong \operatorname{Hom}(M(y^{-1}w_1 \circ \lambda), M^{s_\alpha}(y^{-1}w_2 \circ \lambda)).$$

On the other hand since $y^{-1}w_1 \rightarrow y^{-1}w_2$, the Jantzen sum formula implies that

$$[M(y^{-1}w_2 \circ \lambda) : L(y^{-1}w_1 \circ \lambda)] = 1.$$

Hence by (21) $[M^{s_\alpha}(y^{-1}w_2 \circ \lambda) : L(y^{-1}w_1 \circ \lambda)] = 1$, and therefore,

$$\dim_{\mathbb{C}} \operatorname{Hom}(M^y(w_1 \circ \lambda), M^{ys_\alpha}(w_2 \circ \lambda)) \leq 1.$$

It remains to show that the compositions in (a) and (b) are non-trivial. By the above, this is equivalent to the non-triviality of the compositions

$$\begin{aligned} M(y^{-1}w_1 \circ \lambda) &\rightarrow M(y^{-1}w_2 \circ \lambda) \rightarrow M^{s_\alpha}(y^{-1}w_2 \circ \lambda) \\ \text{and } M(y^{-1}w_1 \circ \lambda) &\rightarrow M^{s_\alpha}(y^{-1}w_1 \circ \lambda) \rightarrow M^{s_\alpha}(y^{-1}w_2 \circ \lambda), \end{aligned}$$

respectively. Therefore we may assume that $y = 1$.

Since $\langle w_2(\lambda + \rho), \alpha^\vee \rangle \in \mathbb{N}$, we have the exact sequence

$$(36) \quad 0 \rightarrow M(s_\alpha w_2 \circ \lambda) \rightarrow M(w_2 \circ \lambda) \rightarrow M^{s_\alpha}(w_2 \circ \lambda) \rightarrow M^{s_\alpha}(s_\alpha w_2 \circ \lambda) \rightarrow 0$$

by Proposition 5.17. On the other hand $w_1 \circ \lambda \not\leq s_\alpha w_2 \circ \lambda$ as we have the square

$$\begin{array}{ccc} & w_1 & \\ \nearrow & & \searrow \\ s_\alpha w_1 & & w_2. \\ \searrow & & \nearrow \\ & s_\alpha w_2 & \end{array}$$

by the assumption and (8). Hence the image of the highest weight vector of $M(w_1 \circ \lambda)$ in $M(w_2 \circ \lambda)$ is not in the kernel of the map $M(w_2 \circ \lambda) \rightarrow M^{s_\alpha}(w_2 \circ \lambda)$. This proves the non-triviality of the composition map in (a). Next we show the non-triviality of the composition in (b). Consider the exact sequence

$$0 \rightarrow M(s_\alpha w_1 \circ \lambda) \rightarrow M(s_\alpha w_2 \circ \lambda) \rightarrow N \rightarrow 0$$

in the category $\mathcal{O}_{[\lambda]}^{\mathfrak{g}}$, where $N = M(s_\alpha w_2 \circ \lambda)/M(s_\alpha w_1 \circ \lambda)$. Applying the functor T_{s_α} we obtain the exact sequence

$$0 \rightarrow \mathcal{L}_1 T_{s_\alpha} N \rightarrow M^{s_\alpha}(w_1 \circ \lambda) \rightarrow M^{s_\alpha}(w_2 \circ \lambda) \rightarrow T_i N \rightarrow 0.$$

By Proposition 5.18, the weights $\mathcal{L}_1 T_{s_\alpha} N$ is contained in weights of N , and hence of $M(s_i w_2 \circ \lambda)$. Therefore the image of the highest weight vector of $M(w_1 \circ \lambda)$ in $M^{s_\alpha}(w_1 \circ \lambda)$ does not belong to the kernel of the map $M^{s_\alpha}(w_1 \circ \lambda) \rightarrow M^{s_\alpha}(w_2 \circ \lambda)$. This completes the proof of (i). (ii) Similarly as above, the problem reduces to the case $y = 1$. By the assumption we have $s_\beta w_1 \xrightarrow{y} w_1$, $s_\beta w_2 \xrightarrow{y} w_2$. Hence $w_1 \not\leq s_\beta w_2$

because otherwise $(w_1, s_\beta w_1, s_\beta w_1, w_2)$ is a y -twisted square. Hence (36) proves the assertion. (iii) Again we may assume that $y = 1$ and the assertion follows from (36). \square

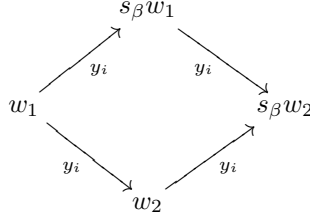
Proof of Proposition 6.6. We prove by induction on i that such an assignment is possible.

As is already remarked the case $i = 0$ is the well-known result of [BGG]. Suppose that $w_1 \xrightarrow{y_i} w_2$ with $i > 1$. Set $\beta = y_{i-1}(\alpha_i) \in \Delta_+^{re}$. The following four cases are possible. (The case $w_1^{-1}(\beta) \in \Delta_+^{re}$, $w_2^{-1}(\beta) \in \Delta_-^{re}$ does not happen by [BGG, Lemma 11.3].)

I) $w_1^{-1}(\beta), w_2^{-1}(\beta) \in \Delta_+^{re}$. In this case $w_1 \xrightarrow{y_{i-1}} w_2$, $\ell^{y_i}(w_1) = \ell^{y_{i-1}}(w_1)$ and $\ell^{y_i}(w_2) = \ell^{y_{i-1}}(w_2)$. By Proposition 6.7 there exists a unique ϵ_{w_1, w_2}^i which makes the digram (35) commutes.

III) $w_1 = s_\beta w_2$. In this case $w_2 \xrightarrow{y_{i-1}} w_1$, $\ell^{y_i}(w_1) = \ell^{y_{i-1}}(w_1) - 2$ and $\ell^{y_i}(w_2) = \ell^{y_{i-1}}(w_2)$. We set $\epsilon_{w_1, w_2}^i = \epsilon_{w_2, w_1}^{i-1}$.

III) $w_1^{-1}(\beta), w_2^{-1}(\beta) \in \Delta_-^{re}$. In this case $w_1 \xrightarrow{y_{i-1}} w_2$, $\ell^{y_i}(w_1) = \ell^{y_{i-1}}(w_1) - 2$ and $\ell^{y_i}(w_2) = \ell^{y_{i-1}}(w_2) - 2$, and we have the following y_i -twisted square.



Note that $\epsilon_{s_\beta w_1, s_\beta w_2}^i$ is defined in I), and $\epsilon_{w_1, s_\beta w_1}^i, \epsilon_{w_2, s_\beta w_2}^i$ are defined in II). We set

$$(37) \quad \epsilon_{w_1, w_2}^i = - \frac{\epsilon_{w_1, s_\beta w_1}^i \epsilon_{s_\beta w_1, s_\beta w_2}^i}{\epsilon_{w_2, s_\beta w_2}^i}.$$

IV) $w_1^{-1}(\beta) \in \Delta_-^{re}$, $w_2^{-1}(\beta) \in \Delta_+^{re}$, $w_2 \neq s_\beta w_1$. In this case there exists a unique $w_3 \in \mathcal{W}$ such that $(s_\beta w_1, w_1, w_3, w_2)$ is a y_i -twisted square. Note that $w_3^{-1}(\beta) \in \Delta_+^{re}$ because $(w_3, w_2, s_\beta w_3, s_\beta w_2)$ is a y_i -twisted square by (8). Since $\epsilon_{s_\beta w_1, w_3}^i$ and ϵ_{w_3, w_2}^i are defined in I), and $\epsilon_{s_\beta w_1, w_1}^i$ is defined in II), we can set

$$(38) \quad \epsilon_{w_1, w_2}^i = - \frac{\epsilon_{s_\beta w_1, w_3}^i \epsilon_{w_3, w_2}^i}{\epsilon_{s_\beta w_1, w_1}^i}.$$

Now let (w_1, w_2, w_3, w_4) be a y_i -twisted square. Set

$$A_i(w_1, w_2, w_3, w_4) = \frac{\epsilon_{w_1, w_2}^i \epsilon_{w_2, w_4}^i}{\epsilon_{w_1, w_3}^i \epsilon_{w_3, w_4}^i}.$$

We need to show that $A_i(w_1, w_2, w_3, w_4) = -1$.

The following four cases are possible.

1) $w_2 = s_\beta w_1$, $w_4 = s_\beta w_3$. In this case the assertion follows from the definition (37).

2) $w_2 = s_\beta w_1$, $w_4 \neq s_\beta w_3$. In this case $(s_\beta w)^{-1}(\beta) \in \Delta_-^{re}$, and $w_4^{-1}(\beta) \in \Delta_+^{re}$ because otherwise $w_3 = s_\beta w_4$. Hence the assertion follows from the definition (38).

3) $w_2 \neq s_\beta w_1$, $w_4 = s_\beta w_3$. In this case $(s_\beta w_1, w_1, s_\beta w_2, w_2)$, $(s_\beta w_1, w_1, s_\beta w_2, w_3)$, $(s_\beta w_2, w_2, s_\beta w_3, w_4)$ are y_i -twisted squares.

$$\begin{array}{ccccccc}
 s_\beta w_1 & \xrightarrow{y_i} & w_1 & \xrightarrow{y_i} & w_2 & & \\
 & \searrow y_i & & \nearrow y_i & & \searrow y_i & \\
 & & s_\beta w_2 & \xrightarrow{y_i} & w_3 & \xrightarrow{y_i} & s_\beta w_3
 \end{array}$$

We have by 1)

$$A_i(s_\beta w_1, w_1, s_\beta w_2, w_2) = A_i(s_\beta w_2, w_2, w_3, s_\beta w_3) = -1$$

and by 2)

$$A_i(s_\beta w_1, w_1, s_\beta w_2, w_3) = -1.$$

But

$$\begin{aligned}
 & A_i(w_1, w_2, w_3, s_\beta w_3) \\
 &= A_i(s_\beta w_1, w_1, s_\beta w_2, w_2) A_i(s_\beta w_2, w_2, w_3, s_\beta w_3) A_i(s_\beta w_1, s_\beta w_2, w_1, w_3).
 \end{aligned}$$

Hence the assertion follows.

4) $w_2 \neq s_\beta w_1$, $w_4 \neq s_\beta w_2$. we see as in [BGG, p.57, c)] that $w_4 \neq s_\beta w_2, s_\beta w_3$, and hence as in [BGG, p.56, 1)] we find that $(s_\beta w_1, s_\beta w_2, s_\beta w_3, s_\beta w_4)$ is also a y_i -twisted square. Hence $w_i^{-1}(\beta) \in \Delta_+^{re}$ for all i or $w_i^{-1}(\beta) \in \Delta_-^{re}$ for all i .

First consider the case $w_i^{-1}(\beta) \in \Delta_+^{re}$ for all i . Then by the definition I) we have the commutative diagram

$$\begin{array}{ccc}
 M^{y_{i-1}}(w_1 \circ \lambda) & \xrightarrow{\epsilon_{w_1, w_a}^{i-1} \epsilon_{w_a, w_4}^{i-1} \varphi_{w_4, w_1}^{\lambda, y_{i-1}}} & M^{y_{i-1}}(w_4 \circ \lambda) \\
 \phi_{w_1}^{y_{i-1}} \downarrow & & \downarrow \phi_{w_4}^{y_{i-1}} \\
 M^y(w_1 \circ \lambda) & \xrightarrow{\epsilon_{w_1, w_a}^i \epsilon_{w_a, w_4}^i \varphi_{w_4, w_1}^{\lambda, y_i}} & M^y(w_4 \circ \lambda)
 \end{array}
 \tag{39}$$

for $a = 2, 3$. Since $\epsilon_{w_1, w_2}^{i-1} \epsilon_{w_2, w_4}^{i-1} = -\epsilon_{w_1, w_3}^{i-1} \epsilon_{w_3, w_4}^{i-1}$ by the induction hypothesis the commutativity of the above diagram implies that $\epsilon_{w_1, w_2}^i \epsilon_{w_2, w_4}^i = -\epsilon_{w_1, w_3}^i \epsilon_{w_3, w_4}^i$ by Proposition 6.7 (ii).

Next consider the case $w_i^{-1}(\beta) \in \Delta_-^{re}$ for all i . Then $(s_\beta w_1, w_1, s_\beta w_2, w_2)$, $(s_\beta w_1, w_1, s_\beta w_3, w_3)$, $(s_\beta w_1, s_\beta w_2, s_\beta w_3, s_\beta w_4)$, $(s_\beta w_2, w_2, s_\beta w_4, w_4)$ and $(s_\beta w_3, w_3, s_\beta w_4, w_4)$ are all y_i -twisted squares. Hence the assertion follows from the equality

$$\begin{aligned}
 & A_i(w_1, w_2, w_3, w_4) A_i(s_\beta w_1, s_\beta w_2, w_1, w_2) A_i(s_\beta w_1, w_1, s_\beta w_3, w_3) \\
 &= A_i(s_\beta w_1, s_\beta w_2, s_\beta w_3, s_\beta w_4) A_i(s_\beta w_2, w_2, s_\beta w_4, w_4) A_i(s_\beta w_3, s_\beta w_4, w_3, w_4).
 \end{aligned}$$

□

Let k be an admissible number, $\lambda \in Pr_k$. Let $y \in \mathcal{W}(\lambda)$, $\{y_i\}$, $\{\phi_w^{y_i}\}$, $\{\epsilon_{w_1, w_2}^i\}$ be as in Proposition 6.6. Because $\{\epsilon_{w_1, w_2}^i\}$ satisfies the condition in Theorem 6.5 there is a corresponding twisted BGG resolution $\mathcal{B}_\bullet^{y_i}(\lambda)$ of $L(\lambda)$ for $i = 0, 1, \dots, l = \ell_\lambda(y)$. Define

$$\Phi_p^{y_{i+1}, y_i} = \bigoplus_{\substack{w \in \mathcal{W}(\lambda) \\ \ell_\lambda^{y_i}(w) = \ell_\lambda^{y_{i+1}}(w) = p}} \phi_w^{y_{i+1}, y_i} : \mathcal{B}_p^{y_i}(w \circ \lambda) \rightarrow \mathcal{B}_p^{y_{i+1}}(w \circ \lambda).$$

Proposition 6.8. *In the above setting $\Phi_{\bullet}^{y_{i+1}, y_i}$ gives a quasi-isomorphism $\mathcal{B}_{\bullet}^{y_i}(\lambda) \sim \mathcal{B}_{\bullet}^{y_{i+1}}(\lambda)$ of complexes for each $i = 0, 2, \dots, l-1$.*

Lemma 6.9. *Let $\lambda \in \mathfrak{h}^*$, y, y_i be as in Proposition 6.6, $w_1, w_2 \in \mathcal{W}(\lambda)$.*

- (i) *Suppose that $w_1 \xrightarrow{y_i} w_2$, $\ell^{y_i}(w_1) = \ell^{y_{i+1}}(w_1)$. Then $w_1 \xrightarrow{y_{i+1}} w_2$.*
- (ii) *Suppose that $w_1 \xrightarrow{y_i} w_2$, $\ell^{y_i}(w_2) = \ell^{y_{i+1}}(w_2)$. Then either of the following two holds.*
 - (a) *$w_2 = s_{\beta} w_1$ and $w_2 \xrightarrow{y_{i+1}} w_1$.*
 - (b) *$w_1 \xrightarrow{y_{i+1}} w_2$.*

Proof. (1) By assumption $s_{\beta} w_1 \xrightarrow{y_i} w_2$. Therefore $(s_{\beta} w_1, w_1, s_{\beta} w_2, w_2)$ is a y_i -twisted square. (2) Similarly, if $w_2 \neq s_{\beta} w_1$ then $(s_{\beta} w_1, w_1, s_{\beta} w_2, w_2)$ y_i -twisted square. The $w_2 \neq s_{\beta} w_1$ case is obvious. \square

Proof of Proposition 6.8. The fact that $\Psi_{\bullet}^{y_i}$ defines a homomorphism of complexes follows from the commutativity of (35), Proposition 6.7 (iii), and Lemma 6.9. Since both complexes are quasi-isomorphic to $L(\lambda)$, to show that it defines a quasi-isomorphism it suffices to check that it defines a non-trivial homomorphism between the corresponding homology spaces. This follows from the fact that $\phi_1^{y_i} : M^{y_i}(\lambda) \rightarrow M^{y_{i+1}}(y)$ sends the highest weight vector of $M^{y_i}(\lambda)$ to the highest weight vector of $M^{y_{i+1}}(\lambda)$. \square

6.5. Two-sided BGG resolutions of G -integrable admissible representations. For $\lambda \in Pr_k$ set

$$\mathcal{W}^i(\lambda) = \{w \in \mathcal{W}(\lambda); \ell_{\lambda}^{\frac{\infty}{2}}(w) = i\}.$$

We note that $\mathcal{W}^i(\lambda)$ is an infinite set unless $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$.

Theorem 6.10. *Let k be an admissible number, $\lambda \in Pr_k^+$*

- (i) *The space $\text{Hom}_{\mathfrak{g}}(W(w \circ \lambda), W(w' \circ \lambda))$ is one-dimensional for $w, w' \in \mathcal{W}(\lambda)$ such that $w \xrightarrow{\frac{\infty}{2}} w'$.*
- (ii) *There exists a complex*

$$C^{\bullet}(\lambda) : \dots \rightarrow C^{-2}(\lambda) \xrightarrow{d_{-2}} C^{-1}(\lambda) \xrightarrow{d_{-1}} C^0(\lambda) \xrightarrow{d_0} C^1(\lambda) \xrightarrow{d_1} C^2(\lambda) \xrightarrow{d_2} \dots$$

in the category \mathcal{O} of the form

$$C^i(\lambda) = \bigoplus_{w \in \mathcal{W}^i(\lambda)} W(w \circ \lambda), \quad d_i = \sum_{\substack{w \in \mathcal{W}^i(\lambda), \\ w \xrightarrow{\frac{\infty}{2}} w' \\ w' \in \mathcal{W}^{i+1}(\lambda)}} d_{w', w},$$

such that

$$H^i(C^{\bullet}(\lambda)) \cong \begin{cases} L(\lambda) & \text{for } i = 0, \\ 0 & \text{for } i \neq 0, \end{cases}$$

where $d_{w', w}$ with $w \xrightarrow{\frac{\infty}{2}} w'$ is a non-trivial \mathfrak{g} -homomorphism $W(w \circ \lambda) \rightarrow W(w' \circ \lambda)$.

Proof. (ii) Let q be the denominator of k and set $M = q\overset{\circ}{Q}^\vee$ if $(r^\vee, q) = 1$ and $M = q\overset{\circ}{Q}$ if $(r^\vee, q) = r^\vee$, so that $\mathcal{W}(\lambda) = \overset{\circ}{\mathcal{W}} \ltimes t_M$. Let β_1, β_2, \dots be a sequence in $\overset{\circ}{P}_+^\vee \cap M$ such that $\beta_i - \beta \in \overset{\circ}{P}_+^\vee \cap M$, $\lim_{i \rightarrow \infty} \alpha(\beta_i) = \infty$ for all $\alpha \in \overset{\circ}{\Delta}_+$.

By Proposition 6.8 there is an inductive system $\{\mathcal{B}_{\bullet}^{-\beta_i}(\lambda)\}$ of twisted BGG resolutions. Let $\mathcal{B}_{-\beta_i}^\bullet(\lambda)$ be the complex $\mathcal{B}_{\bullet}^{-\beta_i}(\lambda)$ with the opposite grading. Thus it is a complex

$$B_{-\beta_i}^\bullet(\lambda) : \cdots \xrightarrow{d_{-2}} B_{\beta_i}^{-1}(\lambda) \xrightarrow{d_{-1}} B_{-\beta_i}^0(\lambda) \xrightarrow{d_0} B_{\beta_i}^1(\lambda) \xrightarrow{d_1} \cdots$$

$$\text{of the form } \mathcal{B}_{\beta_i}^p(\lambda) = \bigoplus_{\substack{w \in \mathcal{W}(\lambda) \\ \ell_\lambda^{-\beta_i}(w) = -p}} M^{-\beta_i}(w \circ \lambda), \quad d_p = \sum_{\substack{w, w' \in \mathcal{W}(\lambda) \\ \ell_\lambda^{-\beta_i}(w) = -p, w \xrightarrow{t_{-\beta_i}} w'}} d_{w', w}^{\beta_i}, \quad d_{w', w}^{\beta_i} :$$

$$M^{-\beta_i}(w \circ \lambda) \rightarrow M^{-\beta_i}(w' \circ \lambda) \text{ such that } H^p(B_{-\beta_i}^\bullet(\lambda)) = \begin{cases} L(\lambda) & \text{if } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$C^\bullet(\lambda) = \varinjlim_i \mathcal{B}_{-\beta_i}^\bullet(\lambda).$$

Then by (25) and Proposition 6.8

$$C^p(\lambda) = \bigoplus_{\substack{w \in \mathcal{W}(\lambda) \\ \ell_\lambda^\infty(w) = p}} \varinjlim_i M^{-\beta_i}(w \circ \lambda) = \bigoplus_{\substack{w \in \mathcal{W}(\lambda) \\ \ell_\lambda^\infty(w) = p}} W(w \circ \lambda) \quad \text{for } p \in \mathbb{Z},$$

$$H^p(C^\bullet(\lambda)) = \varinjlim_i H^p(B_{-\beta_i}^\bullet(\lambda)) = \begin{cases} L(\lambda) & \text{if } p = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and the differential $d_p : C^p(\lambda) \rightarrow C^{p+1}(\lambda)$ has the form

$$d_p = \sum_{\substack{w \in \mathcal{W}^p(\lambda), \quad w' \in \mathcal{W}^{p+1}(\lambda) \\ w \xrightarrow{\frac{\infty}{2}} w'}} d_{w', w},$$

where $d_{w', w} : W(w \circ \lambda) \rightarrow W(w' \circ \lambda)$ is induced by the homomorphisms $d_{w, w'}^{\beta_i} : M^{-\beta_i}(w \circ \lambda) \rightarrow M^{-\beta_i}(w' \circ \lambda)$ with $i = 1, 2, \dots$. To finish the proof of (ii) it remains to show that the map $d_{w', w}$ is nonzero for $w \xrightarrow{\frac{\infty}{2}} w'$.

Let $w', w \in \mathcal{W}(\lambda)$ such that $w' \xrightarrow{\frac{\infty}{2}} w$. We have the commutative diagram

$$\begin{array}{ccc} M^{-\beta_i}(w' \circ \lambda) & \xrightarrow{d_{w, w'}^{\beta_i}} & M^{-\beta_i}(w \circ \lambda) \\ \downarrow \phi_{\beta_i}^{w' \circ \lambda} & & \downarrow \phi_{\beta_i}^{w \circ \lambda} \\ W(w' \circ \lambda) & \xrightarrow{d_{w, w'}} & W(w \circ \lambda) \end{array}$$

for all i . By applying the functor $G_{-\beta_i}$ we obtain the commutative diagram

$$\begin{array}{ccc} M(t_{\beta_i} w' \circ \lambda) & \xrightarrow{G_{\beta_i}(d_{w,w'}^{\beta_i})} & M(t_{\beta_i} w \circ \lambda) \\ \downarrow G_{\beta_i}(\phi_{\beta_i}^{w',\lambda}) & & \downarrow G_{\beta_i}(\phi_{\beta_i}^{w',\lambda}) \\ W(t_{\beta_i} w' \circ \lambda) & \xrightarrow{G_{\beta_i}(d_{w,w'})} & W(t_{\beta_i} w \circ \lambda). \end{array}$$

By Corollary 5.5 $d_{w,w'} \neq 0$ if and only if $G_{\beta_i}(d_{w,w'}) \neq 0$. Therefore it is sufficient to show that $G_{\beta_i}(\phi_{\beta_i}^{w',\lambda}) \circ G_{\beta_i}(d_{w,w'}^{\beta_i}) : M(t_{\beta_i} w' \circ \lambda) \rightarrow W(t_{\beta_i} w \circ \lambda)$ is non-zero for a sufficiently large i .

Write $w' = s_\alpha w$ with $\alpha \in \Delta^{re}$, $\bar{\alpha} \in \Delta_-^\circ$. (This is possible because $s_\alpha = s_{-\alpha}$.) Then, for a sufficiently large i , $\gamma := t_{\beta_i}(\alpha) \in \Delta_+^{re}$ and $t_{\beta_i} s_\alpha w = s_\gamma t_{\beta_i} w \rightarrow t_{\beta_i} w$. The determinant formula [Fre1, Proposition 2 (2)] shows that the image of the highest weight vector of $M(t_{\beta_i} w' \circ \lambda) = M(s_\gamma t_{\beta_i} w \circ \lambda)$ in $M(t_{\beta_i} w \circ \lambda)$ is not in the kernel of the map $G_{\beta_i}(\phi_{\beta_i}^{w',\lambda}) : M(t_{\beta_i} w \circ \lambda) \rightarrow W(t_{\beta_i} w \circ \lambda)$. Therefore $G_{\beta_i}(\phi_{\beta_i}^{w',\lambda}) \circ G_{\beta_i}(d_{w,w'}^{\beta_i})$ is non-zero, and hence so is $d_{w,w'}$.

Finally we shall prove (i). Note that

$$\mathrm{Hom}_{\mathfrak{g}}(W(w' \circ \lambda), W(w \circ \lambda)) = \varprojlim_i \mathrm{Hom}_{\mathfrak{g}}(M^{-\beta_i}(w' \circ \lambda), W(w \circ \lambda))$$

and that $\mathrm{Hom}_{\mathfrak{g}}(M^{-\beta_i}(w' \circ \lambda), W(w \circ \lambda))$ is at most one-dimensional (see the proof of Proposition 6.7). Hence from the proof of (ii) it follows that $\mathrm{Hom}(M^{-\beta_i}(w' \circ \lambda), W(w \circ \lambda)) = \mathbb{C} \phi_{\beta_i}^{w \circ \lambda \circ d_{w,w'}^{\beta_i}}$ for a sufficiently large i and that the map $\mathrm{Hom}(M^{-\beta_j}(w' \circ \lambda), W(w \circ \lambda)) \rightarrow \mathrm{Hom}(M^{-\beta_i}(w' \circ \lambda), W(w \circ \lambda))$ is an isomorphism for sufficiently large $i < j$. This proves that $\mathrm{Hom}_{\mathfrak{g}}(W(w' \circ \lambda), W(w \circ \lambda))$ is one-dimensional. \square

Remark 6.11. By Theorem 6.10 (i) the resolution in Theorem 6.10 (ii) may be described in terms of screening operators as in [BF] provided that the existence of corresponding cycles are established, see e.g. [TK].

The following assertion is an immediate consequence of Theorem 6.10 which generalizes [FF2, Theorem 4.1].

Theorem 6.12. *Let k be an admissible number, $\lambda \in Pr_k^+$. Then*

$$\begin{aligned} H^{\frac{\infty}{2}+p}(\mathfrak{a}, L(\lambda)) &= \bigoplus_{w \in \mathcal{W}^p(\lambda)} \mathbb{C}_{w \circ \lambda} \quad \text{as } \mathfrak{h}\text{-modules,} \\ H^{\frac{\infty}{2}+p}(L\mathfrak{n}, L(\lambda)) &= \bigoplus_{w \in \mathcal{W}^p(\lambda)} \pi_{w \circ \lambda + h \vee \Lambda_0} \quad \text{as } \mathcal{H}\text{-modules.} \end{aligned}$$

6.6. A description of vacuum admissible representation. Let $V^k(\mathring{\mathfrak{g}})$ be the universal affine vertex algebra associated with $\mathring{\mathfrak{g}}$ at level k :

$$V^k(\mathring{\mathfrak{g}}) = U(\mathfrak{g}) \otimes_{U(\mathring{\mathfrak{g}}[t] \oplus \mathbb{C}K)} \mathbb{C}_k,$$

where \mathbb{C}_k is the one-dimensional representations of $\mathring{\mathfrak{g}}[t] \oplus \mathbb{C}K$ on which $\mathring{\mathfrak{g}}[t]$ acts trivially and K acts as the multiplication by k . By [Fre2, Proposition 5.2] we have an injective homomorphism of vertex algebras

$$V^k(\mathring{\mathfrak{g}}) \hookrightarrow W(k\Lambda_0).$$

Hence $V^k(\overset{\circ}{\mathfrak{g}})$ may be regarded as a vertex subalgebra of $W(k\lambda_0)$.

Note that $L(k\Lambda_0)$ is the unique simple quotient of $V^k(\overset{\circ}{\mathfrak{g}})$.

Proposition 6.13. *Let k be an admissible number, $\Phi : W(\dot{s}_0 \circ k\Lambda_0) \rightarrow W(k\Lambda_0)$ a non-zero \mathfrak{g} -homomorphism. Then the image of the highest weight vector of $W(\dot{s}_0 \circ k\Lambda_0)$ generates the maximal submodule of $V^k(\overset{\circ}{\mathfrak{g}}) \subset W(k\Lambda_0)$.*

Proof. By [KW1] the maximal submodule of $V^k(\overset{\circ}{\mathfrak{g}})$ is generated by a singular vector v of weight $\dot{s}_0 \circ k\Lambda_0$. Consider the two-sided resolution $C^\bullet(k\Lambda_0)$ of $L(k\Lambda_0)$ in Theorem 6.10 (ii). Because it is a resolution of $L(k\Lambda_0)$ and $V^k(\overset{\circ}{\mathfrak{g}}) \subset W(k\Lambda_0)$, the vector v must be in the image of $d_{1,w} : W(w \circ k\Lambda_0) \rightarrow W(k\Lambda_0)$ for some $w \in \mathcal{W}^{-1}(k\Lambda_0)$. Since the weight $w \circ k\Lambda_0$ is strictly smaller than $\dot{s}_0 \circ k\Lambda_0$ for $w \in W^{-1}(k\Lambda_0) \setminus \{\dot{s}_0\}$, the only possibility is that v is the image of the highest weight vector of $W(\dot{s}_0 \circ k\Lambda_0)$. The assertion follows since $\dim_{\mathbb{C}} \text{Hom}_{\mathfrak{g}}(W(\dot{s}_0 \circ k\Lambda_0), W(k\Lambda_0)) = 1$ by Theorem 6.10 (i). \square

6.7. Two-sided BGG resolutions of more general admissible representations.

tions. Let $\lambda \in Pr_{k,y}$ with $y = \bar{y}t_\eta$, $\bar{y} \in \overset{\circ}{\mathcal{W}}$, $\eta \in \overset{\circ}{Q}^\vee$. Then there exists $\lambda_1 \in Pr_k^+$ such that $\lambda = y \circ \lambda_1$. Since $y(\Delta(\lambda_1)_+) \subset \Delta_+^{re}$, $T_y : \mathcal{O}_{[\lambda_1]}^{\mathfrak{g}} \rightarrow \mathcal{O}_{[\lambda]}^{\mathfrak{g}}$ is exact,

$$\begin{aligned} T_y L(\lambda_1) &\cong L(\lambda), \\ T_y W(w \circ \lambda_1) &\cong T_y \varinjlim_i M^{-\beta_i}(w \circ \lambda_1) \cong \varinjlim_i T_y M^{-\beta_i}(w \circ \lambda_1) \\ &\cong \varinjlim_i M^{-y(\beta_i)}(ywy^{-1} \circ \lambda) \cong W^{\bar{y}}(ywy^{-1} \circ \lambda) \end{aligned}$$

by (26), Lemmas 5.11 and 5.14, where $(\beta_1, \beta_2, \dots)$ is a sequence in $\overset{\circ}{P}_+^\vee$ such that $\beta_i - \beta_{i-1} \in \overset{\circ}{P}_+^\vee$ and $\varinjlim_i \alpha(\beta_i) = \infty$ for all $\alpha \in \overset{\circ}{\Delta}_+$. Therefore the following assertion follows immediately from Theorem 6.5.

Theorem 6.14. *Let k be an admissible number, $\lambda \in Pr_{k,y}$ with $y = \bar{y}t_\eta$, $\bar{y} \in \overset{\circ}{\mathcal{W}}$, $\eta \in \overset{\circ}{Q}^\vee$. Then there exists a complex*

$$C^\bullet(\lambda) : \dots \xrightarrow{d_{-3}} C^{-2}(\lambda) \xrightarrow{d_{-2}} C^{-1}(\lambda) \xrightarrow{d_{-1}} C^0(\lambda) \xrightarrow{d_0} C^1(\lambda) \xrightarrow{d_1} C^2(\lambda) \xrightarrow{d_2} \dots$$

in the category \mathcal{O} of the form $C^i = \bigoplus_{\substack{w \in \mathcal{W}(\lambda) \\ \ell_{\lambda^{-1}}(w) = i}} W^{\bar{y}}(w \circ \lambda)$, $d_i = \sum_{\substack{w \in \mathcal{W}^i(\lambda), \\ w' \in \mathcal{W}^{i+1}(\lambda) \\ w \xrightarrow{\frac{\alpha}{2}} w'}} d_{w',w}$

such that

$$H^i(C^\bullet(\lambda)) \cong \begin{cases} L(\lambda) & \text{for } i = 0, \\ 0 & \text{for } i \neq 0. \end{cases}$$

7. SEMI-INFINITE RESTRICTION AND INDUCTION

7.1. Feigin-Frenkel parabolic induction. Let $\overset{\circ}{\mathfrak{p}}$ be a parabolic subalgebra containing $\overset{\circ}{\mathfrak{b}}_-$, and let $\overset{\circ}{\mathfrak{p}} = \overset{\circ}{\mathfrak{l}} \oplus \overset{\circ}{\mathfrak{m}}_-$ be the direct sum decomposition of $\overset{\circ}{\mathfrak{p}}$, where $\overset{\circ}{\mathfrak{l}}$ is the Levi subalgebra containing $\overset{\circ}{\mathfrak{h}}$ and $\overset{\circ}{\mathfrak{m}}_-$ is the nilpotent radical of $\overset{\circ}{\mathfrak{p}}$. Denote by

$\mathring{\mathfrak{m}} \subset \mathring{\mathfrak{n}}$ the opposite algebra of $\mathring{\mathfrak{m}}_-$, so that $\mathring{\mathfrak{g}} = \mathring{\mathfrak{p}} \oplus \mathring{\mathfrak{m}}$. Let

$$\mathring{\mathfrak{l}} = \mathring{\mathfrak{l}}_0 \oplus \bigoplus_{i=1}^s \mathring{\mathfrak{l}}_i$$

be the decomposition of $\mathring{\mathfrak{l}}$ into direct sum of simple Lie subalgebras $\mathring{\mathfrak{l}}_i$, $i = 1, \dots, s$, and its center $\mathring{\mathfrak{l}}_0$ of $\mathring{\mathfrak{l}}$. Let $\mathring{\mathfrak{h}}_i = \mathring{\mathfrak{l}} \cap \mathring{\mathfrak{h}}$, the Cartan subalgebra of $\mathring{\mathfrak{l}}_i$, and denote by $\mathring{\Delta}_i \subset \mathring{\Delta}$ be the subroot system of $\mathring{\mathfrak{g}}$ corresponding to $\mathring{\mathfrak{l}}_i$, $\mathring{\Pi}_i = \mathring{\Pi} \cap \mathring{\Delta}_i$. Let h_i^\vee be the dual Coxeter number of $\mathring{\mathfrak{l}}_i$ (with a convention $h_0^\vee = 0$), θ_i the highest root of $\mathring{\Delta}_i$, $\theta_{i,s}$ the highest short roof of $\mathring{\Delta}_i$.

Define the graded subalgebra $\mathfrak{l}_i = \mathring{\mathfrak{l}}_i[t, t^{-1}] \oplus \mathbb{C}K \subset \mathfrak{g}$ for $i = 0, 1, \dots, s$. Set

$$K_i = \frac{2}{(\theta_i | \theta_i)} K,$$

and we consider K_i as an element of $\mathring{\mathfrak{l}}_i$. Thus,

$$\mathfrak{l}_i = \mathring{\mathfrak{l}}_i[t, t^{-1}] \oplus \mathbb{C}K_i,$$

and $\mathfrak{h}_i := \mathring{\mathfrak{h}}_i \oplus \mathbb{C}K_i$ is a Cartan subalgebra of $\mathring{\mathfrak{l}}_i$.

Let

$$\mathfrak{l} = \bigoplus_{i=0}^s \mathfrak{l}_i, \quad \mathfrak{t} = \bigoplus_{i=0}^s \mathfrak{h}_i.$$

Then the grading of \mathfrak{l}_i induces the grading of \mathfrak{l} .

For $k \in \mathbb{C}$ define $k_0, \dots, k_s \in \mathbb{C}$ by

$$k_0 = k + h^\vee, \quad k_i + h_i^\vee = \frac{2}{(\theta_i | \theta_i)} (k + h^\vee) \quad \text{for } i = 1, \dots, s.$$

Lemma 7.1. *Let k be an admissible number for \mathfrak{g} . Then k_i , $i = 1, \dots, s$, is an admissible number for the Kac-Moody algebra \mathfrak{l}_i .*

Let $\mathcal{O}_k^{\mathfrak{g}}$ be the full subcategory of $\mathcal{O}^{\mathfrak{g}}$ consisting of objects on which K acts as the multiplication by k , and let $\mathcal{O}_{(k_0, \dots, k_s)}^{\mathfrak{l}}$ be the full subcategory of $\mathcal{O}^{\mathfrak{l}}$ consisting of objects on which K_i acts as the multiplication by k_i , $i = 0, 1, \dots, s$. Feigin and Frenkel [FF2, 5.2], [Fre2, §6] constructed a functor

$$\text{F-ind}_{\mathfrak{l}}^{\mathfrak{g}} : \mathcal{O}_{(k_0, k_1, \dots, k_s)}^{\mathfrak{l}} \rightarrow \mathcal{O}_k^{\mathfrak{g}}, \quad M \rightarrow \text{F-ind}_{\mathfrak{l}}^{\mathfrak{g}}(M),$$

which enjoys the property

$$(40) \quad \text{F-ind}_{\mathfrak{l}}^{\mathfrak{g}}(M) \cong SSL\mathring{\mathfrak{m}}_{\otimes \mathbb{C}} M$$

as modules over

$$L\mathring{\mathfrak{m}} = \mathring{\mathfrak{m}}[t, t^{-1}] \subset \mathfrak{g}.$$

In particular $\text{F-ind}_{\mathfrak{l}}^{\mathfrak{g}}$ is an exact functor.

Denote by $W_{\mathfrak{l}_i}(\lambda^{(i)})$ of \mathfrak{l}_i the Wakimoto module of the affine Kac-Moody algebra \mathfrak{l}_i with highest weight $\lambda^{(i)} \in \mathfrak{h}_i^*$ and by $L_{\mathfrak{l}}(\lambda^{(i)})$ the irreducible highest weight representation of \mathfrak{l}_i with highest weight $\lambda^{(i)}$ (with a convention that $W_{\mathfrak{l}_0}(\lambda^{(0)})$ is the irreducible representation of the Heisenberg algebra \mathfrak{l}_0 with highest weight $\lambda^{(0)}$).

For $\lambda \in \mathfrak{t}^*$ let $W_{\mathfrak{l}}(\lambda)$ and $L_{\mathfrak{l}}(\lambda)$ be the Wakimoto module and the irreducible highest weight representation of \mathfrak{l} with highest weight λ :

$$W_{\mathfrak{l}}(\lambda) = \bigotimes_{i=0}^s W_{\mathfrak{l}_i}(\lambda|_{\mathfrak{h}_i}), \quad L_{\mathfrak{l}}(\lambda) = \bigotimes_{i=0}^s L_{\mathfrak{l}_i}(\lambda|_{\mathfrak{h}_i}).$$

For $\lambda \in \mathfrak{h}^*$, define $\lambda_{\mathfrak{l}} \in \mathfrak{t}^*$ by

$$\lambda_{\mathfrak{l}}|_{\mathfrak{h}_i} = \lambda|_{\mathfrak{h}_i} \text{ and } (\lambda_{\mathfrak{l}} + \rho_i)(K_i) = \frac{2}{(\theta_i|\theta_i)}(\lambda + \rho)(K)$$

for $i = 0, 1, \dots, s$.

Proposition 7.2 ([FF2]). *For $\lambda \in \mathfrak{h}^*$ we have $\mathrm{F}\text{-ind}_{\mathfrak{p}}^{\mathfrak{g}} W_{\mathfrak{l}}(\lambda_{\mathfrak{l}}) \cong W(\lambda)$.*

Proof. By using the Hochschild-Serre spectral sequence for $L_{\mathfrak{m}}^{\circ} \subset \mathfrak{a}$ we see from (40) that $H^{\frac{\infty}{2}+i}(\mathfrak{a}, \mathrm{F}\text{-ind}_{\mathfrak{l}}^{\mathfrak{g}} W_{\mathfrak{l}}(\lambda_{\mathfrak{l}})) \cong \begin{cases} \mathbb{C} & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases}$ Hence the assertion follows from Theorem 4.6 and the description of the functor $\mathrm{F}\text{-ind}_{\mathfrak{l}}^{\mathfrak{g}}$ in [Fre2]. \square

7.2. Decomposition of integral Weyl groups. Let k be an admissible number with denominator q , $\lambda \in Pr_k^+$. Let $\mathring{\mathcal{W}}_{S_i}$ be the parabolic subgroup of $\mathring{\mathcal{W}}$ corresponding to $\mathring{\mathfrak{l}}_i$, $\mathring{\mathcal{W}}_S = \mathring{\mathcal{W}}_{S_1} \times \mathring{\mathcal{W}}_{S_2} \times \cdots \times \mathring{\mathcal{W}}_{S_s}$. Define $\mathring{\alpha}_0^{(i)} \in \Delta(\lambda)$, $i = 1, \dots, s$, by

$$\mathring{\alpha}_0^{(i)} = \begin{cases} \theta_i + q\delta & \text{if } \theta_i \in \mathring{\Delta}_{long}, r_i^{\vee} \nmid q \\ \theta_{i,s} + \frac{q}{r_i^{\vee}}\delta & \text{if } \theta_i \in \mathring{\Delta}_{long}, r_i^{\vee} | q \\ \theta_i + q\delta & \text{if } \theta_i \in \mathring{\Delta}_{short}, r^{\vee} \nmid q \\ \theta_i + \frac{q}{r^{\vee}}\delta & \text{if } \theta_i \in \mathring{\Delta}_{short}, r^{\vee} | q, \end{cases}$$

where and r_i^{\vee} is the lacing number of \mathfrak{l}_i . Set $\mathring{s}_0^{(i)} = s_{\mathring{\alpha}_0^{(i)}}$.

Let $\mathcal{W}(\lambda)_{S_i}$ be the subgroup of $\mathcal{W}(\lambda)$ generated by $\mathring{\mathcal{W}}_{S_i}$ and $\mathring{s}_0^{(i)}$. Then

$$\mathcal{W}(\lambda)_S = \mathcal{W}(\lambda)_{S_1} \times \mathcal{W}(\lambda)_{S_2} \times \cdots \times \mathcal{W}(\lambda)_{S_s}$$

is the subgroup corresponding to $\mathring{\mathcal{W}}_S$ described in §3.4. Let $\mathcal{W}(\lambda)^S \subset \mathcal{W}(\lambda)$ such that

(41)

$$\mathcal{W}(\lambda) = \mathcal{W}(\lambda)_S \times \mathcal{W}(\lambda)^S, \quad \ell_{\lambda}^{\infty}(uv) = \ell_{\lambda}^{\frac{\infty}{2}}(u) + \ell_{\lambda}^{\frac{\infty}{2}}(v) \text{ for } u \in \mathcal{W}(\lambda), v \in \mathcal{W}(\lambda)^S$$

as in Theorem 3.3.

Let $w, w' \in \mathcal{W}(\lambda)_{S_i} \subset \mathcal{W}(\lambda)$ such that $w \xrightarrow{\frac{\infty}{2}} w'$. Then $w \circ_{\mathfrak{l}_i} \lambda_{\mathfrak{l}}^{(i)} = (w \circ \lambda)_{\mathfrak{l}}^{(i)}$,

where $\circ_{\mathfrak{l}_i}$ is the dot action of $\mathcal{W}(\lambda)_{S_i}$ on \mathfrak{h}_i^* and $\lambda_{\mathfrak{l}_i}^{(i)} = \lambda|_{\mathfrak{h}_i}$. Let, for any $w \in \mathcal{W}(\lambda)$, $w \circ_{\mathfrak{l}_i} \lambda_{\mathfrak{l}} \in \mathfrak{t}^*$ be a weight such that $w \circ_{\mathfrak{l}_i} \lambda_{\mathfrak{l}}|_{\mathfrak{l}_i} = w \circ_{\mathfrak{l}_i} \lambda_{\mathfrak{l}}^{(i)}$ and $w \circ_{\mathfrak{l}_i} \lambda_{\mathfrak{l}}|_{\mathfrak{l}_j} = \lambda_{\mathfrak{l}}^{(j)}$ for $j \neq i$. Then $\Phi \in \mathrm{Hom}_{\mathfrak{l}_i}(W_{\mathfrak{l}_i}(w \circ_{\mathfrak{l}_i} \lambda_{\mathfrak{l}}^{(i)}), W_{\mathfrak{l}_i}(w' \circ_{\mathfrak{l}_i} \lambda_{\mathfrak{l}}^{(i)}))$ induces an \mathfrak{l} -homomorphism $\tilde{\Phi} : W_{\mathfrak{l}}(w \circ_{\mathfrak{l}_i} \lambda) \rightarrow W_{\mathfrak{l}}(w' \circ_{\mathfrak{l}_i} \lambda)$. By applying the functor $\mathrm{F}\text{-ind}_{\mathfrak{l}}^{\mathfrak{g}}$ we obtain a \mathfrak{g} -homomorphism $\mathrm{F}\text{-ind}_{\mathfrak{l}}^{\mathfrak{g}}(\tilde{\Phi}) : W(w \circ \lambda) \rightarrow W(w' \circ \lambda)$ by Proposition 7.2.

Proposition 7.3. *Let $\lambda \in Pr_k^+$, $w, w' \in \mathcal{W}(\lambda)_{S_i}$ such that $w \xrightarrow{\frac{\infty}{2}} w'$ with $i \in \{1, 2, \dots, s\}$. Then the above described correspondence $\Phi \mapsto \text{F-ind}_I^{\mathfrak{g}}(\tilde{\Phi})$ defines a linear isomorphism*

$$\text{Hom}_{\mathfrak{t}_i}(W_{\mathfrak{t}_i}(w \circ_{\mathfrak{t}_i} \lambda_I^{(i)}), W_{\mathfrak{t}_i}(w' \circ_{\mathfrak{t}_i} \lambda_I^{(i)})) \xrightarrow{\sim} \text{Hom}_{\mathfrak{g}}(W(w \circ \lambda), W(w' \circ \lambda)).$$

Proof. By Theorem 6.5 both $\text{Hom}_{\mathfrak{t}_i}(W_{\mathfrak{t}_i}(w \circ_{\mathfrak{t}_i} \lambda_I^{(i)}), W_{\mathfrak{t}_i}(w' \circ_{\mathfrak{t}_i} \lambda_I^{(i)}))$ and $\text{Hom}_{\mathfrak{g}}(W(w \circ \lambda), W(w' \circ \lambda))$ is one-dimensional. But the correspondence $\Phi \mapsto \text{F-ind}_I^{\mathfrak{g}}(\tilde{\Phi})$ is clearly injective. \square

7.3. Semi-infinite restriction functors. Let $M \in \mathcal{O}_k^{\mathfrak{g}}$. Then $H^{\frac{\infty}{2}+p}(L\mathfrak{m}^{\circ}, M)$, $p \in \mathbb{Z}$, is an \mathfrak{l} -module on which K_i acts as the multiplication by k_i , see e.g. [HT, Proposition 2.3]. Hence

$$\text{S-res}_I^{\mathfrak{g}} := H^{\frac{\infty}{2}+0}(L\mathfrak{m}^{\circ}, ?)$$

defines a functor $\mathcal{O}_k^{\mathfrak{g}} \rightarrow \mathcal{O}_{(k_0, k_1, \dots, k_s)}^{\mathfrak{l}}$.

Proposition 7.2 gives the following assertion.

Proposition 7.4. *For $\lambda \in \mathfrak{h}^*$ we have $H^{\frac{\infty}{2}+i}(L\mathfrak{m}^{\circ}, W(\lambda)) = 0$ for $i \neq 0$ and*

$$\text{S-res}_I^{\mathfrak{g}} W(\lambda) \cong W_I(\lambda_I).$$

For $f \in \text{Hom}_I(W_I(w \circ_{\mathfrak{t}_i} \lambda_I), W_I(w' \circ_{\mathfrak{t}_i} \lambda_I))$ let f_i denote its restriction to the i -th component $W_{\mathfrak{t}_i}(w \circ_{\mathfrak{t}_i} \lambda_I^{(i)})$, that is, $f_i(m) = f(v_{w \circ_{\mathfrak{t}_i} \lambda_I^{(1)}} \otimes \dots \otimes \overset{i}{m} \otimes \dots \otimes v_{w \circ_{\mathfrak{t}_i} \lambda_I^{(s)}})$, where $v_{w \circ_{\mathfrak{t}_i} \lambda_I^{(i)}}$ is the highest weight vector of $W_{\mathfrak{t}_i}((w \circ_{\mathfrak{t}_i} \lambda_I)^{(i)})$.

Proposition 7.5. *Let $\lambda \in Pr_k^+$ and let $w, w' \in \mathcal{W}(\lambda)_{S_i}$ such that $w \xrightarrow{\frac{\infty}{2}} w'$ as in §7.2. Then $\Psi \mapsto \text{S-res}_I^{\mathfrak{g}}(\Psi)_i$ gives the linear isomorphism*

$$\text{Hom}_{\mathfrak{g}}(W(w \circ \lambda), W(w' \circ \lambda)) \xrightarrow{\sim} \text{Hom}_{\mathfrak{t}_i}(W_{\mathfrak{t}_i}(w \circ_{\mathfrak{t}_i} \lambda_I^{(i)}), W_{\mathfrak{t}_i}(w' \circ_{\mathfrak{t}_i} \lambda_I^{(i)})).$$

Proof. The assertion follows by observing that $\text{S-res}_I^{\mathfrak{g}}(\text{F-ind}_I^{\mathfrak{g}}(\tilde{\Phi})) = \tilde{\Phi}$ for $\Phi \in \text{Hom}_{\mathfrak{t}_i}(W_{\mathfrak{t}_i}(w \circ_{\mathfrak{t}_i} \lambda_I^{(i)}), W_{\mathfrak{t}_i}(w' \circ_{\mathfrak{t}_i} \lambda_I^{(i)}))$ in the notation of §7.2. \square

7.4. Semi-infinite restriction of admissible affine vertex algebras. The space $\text{S-res}_I^{\mathfrak{g}}(V^k(\mathfrak{g}^{\circ}))$ inherits a vertex algebra structure from $V^k(\mathfrak{g}^{\circ})$ and we have an vertex algebra homomorphism

$$\bigotimes_{i=0}^s V^{k_i}(\mathfrak{l}_i^{\circ}) \rightarrow \text{S-res}_I^{\mathfrak{g}}(V^k(\mathfrak{g}^{\circ})),$$

where $V^{k_i}(\mathfrak{l}_i^{\circ})$ denote the universal affine vertex algebra associated with \mathfrak{l}_i at level k_i . By composing with the map $\text{S-res}_I^{\mathfrak{g}}(V^k(\mathfrak{g}^{\circ})) \rightarrow \text{S-res}_I^{\mathfrak{g}}(L(k\Lambda_0))$ induced by the surjection $V^k(\mathfrak{g}^{\circ}) \twoheadrightarrow L(k\Lambda_0)$ this gives rise to a vertex algebra homomorphism

$$(42) \quad \bigotimes_{i=0}^s V^{k_i}(\mathfrak{l}_i) \rightarrow \text{S-res}_I^{\mathfrak{g}}(L(k\Lambda_0)).$$

Theorem 7.6. *Let k be an admissible number. The vertex algebra homomorphism (42) factors through the vertex algebra homomorphism*

$$\bigotimes_{i=0}^s L_{\mathfrak{l}_i}(k_i \Lambda_0) \hookrightarrow \text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(L(\mathring{\mathfrak{m}}_i), L(k\Lambda_0)).$$

Proof. Put $\lambda = k\Lambda_0$ and let $C^\bullet(\lambda)$ be the two-sided resolution of $L(k\Lambda_0)$ in Theorem 6.10. By the vanishing assertion of Proposition 7.4 the semi-infinite cohomology $H^{\frac{\infty}{2}+\bullet}(L(\mathring{\mathfrak{m}}_i), L(\lambda))$ is isomorphic to the complex $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(C^\bullet(\lambda))$ obtained from $C^\bullet(\lambda)$ applying the functor $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}$.

Consider the map $C^{-1}(\lambda) \supset W(\dot{s}_0^{(i)} \circ \lambda) \xrightarrow{d_{\dot{s}_0^{(i)}, 1}} W(\lambda) \subset C^0(\lambda)$ for $i = 1, \dots, s$. By applying the functor $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}$ this induces a non-zero homomorphism $W_{\mathfrak{l}}(\dot{s}_0^{(i)} \circ_{\mathfrak{l}_i} \lambda_{\mathfrak{l}}) \rightarrow W_{\mathfrak{l}}(\lambda_{\mathfrak{l}})$ by Proposition 7.5, and the image of the highest weight vector of $W_{\mathfrak{l}}(\dot{s}_0^{(i)} \circ_{\mathfrak{l}_i} \lambda_{\mathfrak{l}})$ generates the maximal \mathfrak{l}_i -submodule of $V^{k_i}(\mathfrak{l}_i) \subset W_{\mathfrak{l}}(\lambda_{\mathfrak{l}})$ by Proposition 6.13. It follows that the maximal \mathfrak{l} -submodule of $\bigotimes_{i=0}^s V^{k_i}(\mathfrak{l}_i) \subset W_{\mathfrak{l}}(\lambda)$ is in the image of $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(d_{-1}) : \text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(C^{-1}(\lambda)) \rightarrow \text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(C^0(\lambda))$. This completes the proof. \square

7.5. The case of minimal parabolic subalgebras. Consider the case that $\mathring{\mathfrak{p}}$ is generated by $\mathring{\mathfrak{b}}_-$ and e_j with $j = 1, \dots, \ell$. Then $\mathring{\mathfrak{l}} = \mathring{\mathfrak{l}}_0 \oplus \mathring{\mathfrak{l}}_1$, $\mathfrak{l}_1 = \mathfrak{sl}_2^{(j)}$ and $\mathfrak{l}_1 = \widehat{\mathfrak{sl}}_2^{(j)}$.

Theorem 7.7 ($\mathring{\mathfrak{p}}$ minimal). *Let k be an admissible number and let M be a module over the vertex algebra $L(k\Lambda_0)$. Then, for each $p \in \mathbb{Z}$, $H^{\frac{\infty}{2}+p}(L(\mathring{\mathfrak{m}}), M)$ is a direct sum of admissible representations of level k_1 as $\widehat{\mathfrak{sl}}_2^{(j)}$ -modules.*

Proof. By Theorem 7.6, $L_{\mathfrak{l}_1}(k_1\Lambda_0)$ is a vertex subalgebra of $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(L(k\Lambda_0)) = H^{\frac{\infty}{2}+0}(L(\mathring{\mathfrak{m}}), L(k\Lambda_0))$. If M is a module over $L(k\Lambda_0)$ then $H^{\frac{\infty}{2}+p}(L(\mathring{\mathfrak{m}}), M)$ is a module over $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(L(k\Lambda_0))$, and therefore it is a module over $L_{\mathfrak{l}_1}(k_1\Lambda_0)$. On the other hand it is known by [AM] that any module over $L_{\mathfrak{l}_1}(k_1\Lambda_0)$ in the category $\mathcal{O}^{\mathfrak{l}_1}$ must be a direct sum of admissible representations of $\mathfrak{l}_1 \cong \widehat{\mathfrak{sl}}_2$. The assertion follows since $H^{\frac{\infty}{2}+p}(L(\mathring{\mathfrak{m}}), M)$ is an injective limit of objects in $\mathcal{O}^{\mathfrak{l}_1}$. \square

The following assertion generalizes [HT, Theorem 3.8] in the case that $\mathring{\mathfrak{p}}$ is minimal.

Theorem 7.8 ($\mathring{\mathfrak{p}}$ minimal). *Let k be an admissible number, $\lambda \in Pr_k^+$. Then*

$$H^{\frac{\infty}{2}+p}(L(\mathring{\mathfrak{m}}), L(\lambda)) \cong \bigoplus_{\substack{w \in \mathcal{W}(\lambda)^S \\ \ell_{\lambda}^{\frac{\infty}{2}}(w) = p}} L_{\mathfrak{l}}((w \circ \lambda)_{\mathfrak{l}})$$

as \mathfrak{l} -modules.

Proof. It is known by [FM] that $L(\lambda)$ with $\lambda \in Pr_k^+$ is a module over $L(k\Lambda_0)$. Therefore $H^{\frac{\infty}{2}+\bullet}(L(\mathring{\mathfrak{m}}), L(\lambda))$ is a direct sum of irreducible admissible representations as $\widehat{\mathfrak{sl}}_2^{(j)}$ -modules by Theorem 7.7. Also, as is remarked in the proof of Proposition 7.6 $H^{\frac{\infty}{2}+\bullet}(L(\mathring{\mathfrak{m}}), L(\lambda))$ is the cohomology of the complex $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(C^\bullet(\lambda))$ and we have $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(C^p(\lambda)) = \bigoplus_{w \in \mathcal{W}^p(\lambda)} W_{\mathfrak{l}}((w \circ \lambda)_{\mathfrak{l}})$ by Proposition 7.4.. Now Theorem 3.3

implies that

$$\begin{aligned} & \{(w \circ \lambda)_\mathfrak{l}; w \in \mathcal{W}(\lambda), (w \circ \lambda)_\mathfrak{l}^{(1)}|_{\mathfrak{h}_1} \text{ is a dominant weight for } \widehat{\mathfrak{sl}}_2^{(j)}\} \\ &= \{(w \circ \lambda)_\mathfrak{l}; w \in \mathcal{W}(\lambda), (w \circ \lambda)_\mathfrak{l}^{(1)}|_{\mathfrak{h}_1} \text{ is an admissible weight for } \widehat{\mathfrak{sl}}_2^{(j)}\} \\ &= \{(w \circ \lambda)_\mathfrak{l}; w \in \mathcal{W}(\lambda)^S\}. \end{aligned}$$

Therefore $\{[v_{(w \circ \lambda)_\mathfrak{l}}]; w \in \mathcal{W}(\lambda)^S\}$ forms a basis of $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}, L(\lambda))^{\mathfrak{l}+}$, where $[v_{(w \circ \lambda)_\mathfrak{l}}]$ is the image of the highest weight vector $v_{(w \circ \lambda)_\mathfrak{l}}$ of $W_\mathfrak{l}((w \circ \lambda)_\mathfrak{l})$. This completes the proof. \square

Remark 7.9. In the subsequent paper [A6] we prove that for an admissible number k any $L(k\Lambda_0)$ -module in the category \mathcal{O}^g must be a direct sum of admissible representations. Hence it follows from the proof that the assertion of Theorem 7.8 is valid for any parabolic subalgebra of \mathfrak{g} .

REFERENCES

- [AG] S. Arkhipov and D. Gaitsgory. Differential operators on the loop group via chiral algebras. *Int. Math. Res. Not.*, (4):165–210, 2002.
- [AL] H. H. Andersen and N. Lauritzen. Twisted Verma modules. In *Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000)*, volume 210 of *Progr. Math.*, pages 1–26. Birkhäuser Boston, Boston, MA, 2003.
- [AM] Dražen Adamović and Antun Milas. Vertex operator algebras associated to modular invariant representations for $A_1^{(1)}$. *Math. Res. Lett.*, 2(5):563–575, 1995.
- [A1] T. Arakawa. Vanishing of cohomology associated to quantized Drinfeld-Sokolov reduction. *Int. Math. Res. Not.*, (15):730–767, 2004.
- [A2] T. Arakawa. Representation theory of superconformal algebras and the Kac-Roan-Wakimoto conjecture. *Duke Math. J.*, 130(3):435–478, 2005.
- [A3] T. Arakawa. Representation theory of W -algebras. *Invent. Math.*, 169(2):219–320, 2007.
- [A4] T. Arakawa. Representation theory of W -algebras, II. In *Exploring new structures and natural constructions in mathematical physics*, volume 61 of *Adv. Stud. Pure Math.*, pages 51–90. Math. Soc. Japan, Tokyo, 2011.
- [A5] T. Arakawa. Associated varieties of modules over Kac-Moody algebras and C_2 -cofiniteness of W -algebras. *preprint*, arXiv:1004.1554v2.
- [A6] T. Arakawa. Rationality of admissible affine vertex algebras in the category \mathcal{O} . *preprint*, 2012.
- [Ark1] Sergey Arkhipov. A new construction of the semi-infinite BGG resolution. *preprint*, 1996. math.QA/9605043.
- [Ark2] S. M. Arkhipov. Semi-infinite cohomology of associative algebras and bar duality. *Internat. Math. Res. Notices*, (17):833–863, 1997.
- [Ark3] Sergey Arkhipov. Algebraic construction of contragredient quasi-Verma modules in positive characteristic. In *Representation theory of algebraic groups and quantum groups*, volume 40 of *Adv. Stud. Pure Math.*, pages 27–68. Math. Soc. Japan, Tokyo, 2004.
- [AS] Henning Haahr Andersen and Catharina Stroppel. Twisting functors on \mathcal{O} . *Represent. Theory*, 7:681–699 (electronic), 2003.
- [BF] D. Bernard and G. Felder. Fock representations and BRST cohomology in $SL(2)$ current algebra. *Comm. Math. Phys.*, 127(1):145–168, 1990.
- [BGG] I. N. Bernšteĭn, I. M. Gel’fand, and S. I. Gel’fand. Differential operators on the base affine space and a study of \mathfrak{g} -modules. In *Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971)*, pages 21–64. Halsted, New York, 1975.
- [Feĭ] B. L. Feĭgin. Semi-infinite homology of Lie, Kac-Moody and Virasoro algebras. *Uspekhi Mat. Nauk*, 39(2(236)):195–196, 1984.
- [FF1] B. L. Feĭgin and È. V. Frenkel’. A family of representations of affine Lie algebras. *Uspekhi Mat. Nauk*, 43(5(263)):227–228, 1988.

- [FF2] Boris L. Feigin and Edward V. Frenkel. Affine Kac-Moody algebras and semi-infinite flag manifolds. *Comm. Math. Phys.*, 128(1):161–189, 1990.
- [FF3] Boris L. Feigin and Edward V. Frenkel. Representations of affine Kac-Moody algebras and bosonization. In *Physics and mathematics of strings*, pages 271–316. World Sci. Publ., Teaneck, NJ, 1990.
- [Fie] Peter Fiebig. The combinatorics of category \mathcal{O} over symmetrizable Kac-Moody algebras. *Transform. Groups*, 11(1):29–49, 2006.
- [FKW] Edward Frenkel, Victor Kac, and Minoru Wakimoto. Characters and fusion rules for W -algebras via quantized Drinfel’d-Sokolov reduction. *Comm. Math. Phys.*, 147(2):295–328, 1992.
- [FM] Igor Frenkel and Fyodor Malikov. Kazhdan-Lusztig tensoring and Harish-Chandra categories. *preprint*, 1997. arXiv:q-alg/9703010.
- [Fre1] Edward Frenkel. Determinant formulas for the free field representations of the Virasoro and Kac-Moody algebras. *Phys. Lett. B*, 286(1-2):71–77, 1992.
- [Fre2] Edward Frenkel. Wakimoto modules, opers and the center at the critical level. *Adv. Math.*, 195(2):297–404, 2005.
- [GL] Howard Garland and James Lepowsky. Lie algebra homology and the Macdonald-Kac formulas. *Invent. Math.*, 34(1):37–76, 1976.
- [HT] Shinobu Hosono and Akihiro Tsuchiya. Lie algebra cohomology and $N = 2$ SCFT based on the GKO construction. *Comm. Math. Phys.*, 136(3):451–486, 1991.
- [Kos] Bertram Kostant. Lie algebra cohomology and the generalized Borel-Weil theorem. *Ann. of Math. (2)*, 74:329–387, 1961.
- [KRW] Victor Kac, Shi-Shyr Roan, and Minoru Wakimoto. Quantum reduction for affine superalgebras. *Comm. Math. Phys.*, 241(2-3):307–342, 2003.
- [KT] Masaki Kashiwara and Toshiyuki Tanisaki. Kazhdan-Lusztig conjecture for symmetrizable Kac-Moody Lie algebras. III. Positive rational case. *Asian J. Math.*, 2(4):779–832, 1998. Mikio Sato: a great Japanese mathematician of the twentieth century.
- [KW1] Victor G. Kac and Minoru Wakimoto. Modular invariant representations of infinite-dimensional Lie algebras and superalgebras. *Proc. Nat. Acad. Sci. U.S.A.*, 85(14):4956–4960, 1988.
- [KW2] V. G. Kac and M. Wakimoto. Classification of modular invariant representations of affine algebras. In *Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988)*, volume 7 of *Adv. Ser. Math. Phys.*, pages 138–177. World Sci. Publ., Teaneck, NJ, 1989.
- [KW3] Victor G. Kac and Minoru Wakimoto. On rationality of W -algebras. *Transform. Groups*, 13(3-4):671–713, 2008.
- [Lus] George Lusztig. Hecke algebras and Jantzen’s generic decomposition patterns. *Adv. in Math.*, 37(2):121–164, 1980.
- [RCW] Alvany Rocha-Caridi and Nolan R. Wallach. Projective modules over graded Lie algebras. I. *Math. Z.*, 180(2):151–177, 1982.
- [Soe1] Wolfgang Soergel. Kazhdan-Lusztig polynomials and a combinatoric[s] for tilting modules. *Represent. Theory*, 1:83–114 (electronic), 1997.
- [Soe2] Wolfgang Soergel. Character formulas for tilting modules over Kac-Moody algebras. *Represent. Theory*, 2:432–448 (electronic), 1998.
- [TK] Akihiro Tsuchiya and Yukihiro Kanie. Fock space representations of the Virasoro algebra. Intertwining operators. *Publ. Res. Inst. Math. Sci.*, 22(2):259–327, 1986.
- [Vor1] Alexander A. Voronov. Semi-infinite homological algebra. *Invent. Math.*, 113(1):103–146, 1993.
- [Vor2] Alexander A. Voronov. Semi-infinite induction and Wakimoto modules. *Amer. J. Math.*, 121(5):1079–1094, 1999.
- [Wak] Minoru Wakimoto. Fock representations of the affine Lie algebra $A_1^{(1)}$. *Comm. Math. Phys.*, 104(4):605–609, 1986.

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502 JAPAN

E-mail address: arakawa@kurims.kyoto-u.ac.jp